Generating functionals for harmonic expectation values of paths with fixed end points: Feynman diagrams for nonpolynomial interactions

Hagen Kleinert, Axel Pelster, and Michael Bachmann
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
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We introduce a general class of generating functionals for the calculation of quantum-mechanical expectation values of arbitrary functionals of fluctuating paths with fixed end points in configuration or momentum space. The generating functionals are calculated explicitly for the harmonic oscillator with time-dependent frequency, and used to derive a smearing formula for correlation functions of polynomial and nonpolynomial functions of time-dependent positions and momenta. This formula summarizes the effect of quantum fluctuations, and serves to derive generalized Wick rules and Feynman diagrams for perturbation expansions of nonpolynomial interactions. [S1063-651X(99)04508-0]

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I. INTRODUCTION

A useful technique for describing compactly the properties of a quantum-mechanical system is to define a suitable generating functional of some external source or current \( j(t) \). The desired properties are obtained from functional derivatives with respect to \( j(t) \). For example, the correlation functions and the time evolution amplitude in one space dimension \( x \) are determined by a generating functional which is a path integral in configuration space over all paths \( x(t) \) with fixed end points \( x(t_a) = x_a, \ x(t_b) = x_b \) [[1], Chap. 2]:

\[
(x_b, t_b | x_a, t_a) [j(t)] = \int_{x_a, t_a}^{x_b, t_b} \mathcal{D}x(t) \exp \left( \frac{i}{\hbar} \mathcal{A}[x(t); j(t)] \right), \tag{1.1}
\]

where the exponent contains the classical action \( \mathcal{A}[x(t)] \) plus a source term linear in \( x(t) \):

\[
\mathcal{A}[x(t); j(t)] = \mathcal{A}[x(t)] + \int_{t_a}^{t_b} dt \ x(t) j(t). \tag{1.2}
\]

In this paper we set up a useful alternative expression for the generating functional (1.1) and a related one in momentum space. This alternative expression is obtained by extending the current \( j(t) \) by singular sources proportional to \( \delta(t_b - t) \) and \( \delta(t - t_a) \), and by reducing the path integral (1.1) with fixed end points in configuration space to one with vanishing end points. This will permit us to simplify considerably the calculation of quantum-mechanical correlation functions. To see this simplification explicitly, consider a harmonic oscillator whose action reads

\[
\mathcal{A}[x(t)] = \int_{t_a}^{t_b} dt \left( \frac{M}{2} \dot{x}^2(t) - \frac{1}{2} M \omega^2 x^2(t) \right), \tag{1.3}
\]

for which the generating functional can be calculated as follows [[1], Eq. (3.89)]:

\[
(x_b, t_b | x_a, t_a) [j(t)] = \left( \frac{M \omega}{2 \pi \hbar \sin \omega(t_b - t_a)} \right)^{1/2} \exp \left( i \frac{M \omega}{2 \hbar \sin \omega(t_b - t_a)} \left[ (x_b^2 + x_a^2) \cos \omega(t_b - t_a) - 2 x_a x_b \right] \right) \times \exp \left( \frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{x_a \sin \omega(t_b - t_a) + x_b \sin \omega(t - t_a)}{\sin \omega(t_b - t_a)} j(t) \right) \times \exp \left( - \frac{i}{\hbar M \omega} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \frac{\sin \omega(t_b - t) \sin \omega(t' - t_a)}{\sin \omega(t_b - t_a)} j(t) j(t') \right). \tag{1.4}
\]

The nonzero end points \( x_a \) and \( x_b \) make this expression quite involved. For vanishing end points, however, it simplifies to

\[
(X_b = 0, t_b = 0, t_a) [j(t)] = \left( \frac{M \omega}{2 \pi \hbar \sin \omega(t_b - t_a)} \right)^{1/2} \times \exp \left( - \frac{i}{\hbar M \omega} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \frac{\sin \omega(t_b - t) \sin \omega(t' - t_a)}{\sin \omega(t_b - t_a)} j(t) j(t') \right). \tag{1.5}
\]
The observation which motivates the present paper relies on replacing the current \( j(t) \) in the simple expression (1.5) by

\[
j'(t) = j(t) + M x_a \delta(t-t_a) + M x_b \delta(t_b - t),
\]

(1.6)

where the \( \delta \) functions are understood as \( \delta(t-t_a+\epsilon) \) and \( \delta(t_b - t - \epsilon) \) in the limit \( \epsilon \to 0 \). By performing some partial integrations, this replacement reproduces all terms in the complicated generating functional (1.4), except for a rather trivial additional singular phase factor. The important relation is

\[
(x_b t_b|x_a t_a)[j(t)] = (x_b = 0 \ t_b = 0 \ t_a)
\]

\[
\times [j(t) + M x_a \delta(t-t_a) + M x_b \delta(t_b - t)]
\]

\[
\times \exp \left( \frac{i M}{2 \hbar} (x_b^2 + x_a^2) \delta(0) \right).
\]

(1.7)

In Sec. II we prove that the relation (1.7) holds for an arbitrary quantum-mechanical system whose Hamiltonian has the standard form

\[
H_0(p, x, t) = \frac{p^2}{2M} + V(x, t).
\]

(1.8)

In Sec. III we calculate explicit amplitudes for a harmonic oscillator with arbitrary time-dependent frequency, and as an important application we derive in Sec. IV a smearing formula for calculating expectation values of polynomial and nonpolynomial functions of time-dependent positions and momenta. In particular, this result would allow us to calculate expectation values appearing in perturbation expansions for nonlinear interactions, as, for example, for the nonlinear \( \sigma \) model. In Sec. V we show that our smearing formula generalizes Wick rules and Feynman diagrams for harmonic expectation values from products of variables to mixtures of nonpolynomial functions and polynomials. In Sec. VI, we finally specialize our generating functional to periodic paths.

**II. GENERATING FUNCTIONALS**

We begin by setting up phase-space path integrals for generating functionals with fixed end points in either configuration or momentum space. The action contains additional currents \( k(t) \) and \( j(t) \) coupled linearly to momentum \( p(t) \) and position \( x(t) \). By extending the currents with singular \( \delta \) functions as in Eq. (1.6), we reduce the path integrals with fixed end points to those with vanishing end points. Our procedure applies to arbitrary Hamiltonians \( H_0(p, x, t) \), with certain simplifications resulting from a standard Hamiltonian (1.8).

### A. General phase-space formulation

Consider a quantum-mechanical particle coupled to a momenta and a position source \( k(t) \) and \( j(t) \) with the classical Hamiltonian

\[
H(p, x, t) = H_0(p, x, t) - pk(t) - xj(t),
\]

(2.1)

where the corresponding action reads

\[
A[p(t),x(t);k(t),j(t)] = \int_{t_a}^{t_b} dt \{p(t)\dot{x}(t) - H(p(t),x(t),t)}.
\]

(2.2)

The total time evolution amplitude between fixed space points \( x_a \) and \( x_b \) is given by the path integral

\[
(x_b t_b|x_a t_a)[k(t),j(t)] = \int_{x_a t_a}^{x_b t_b} Dp(t) Dx(t) \left[ \frac{2\pi\hbar}{i} \right] \exp \left( \frac{i}{\hbar} A[p(t),x(t);k(t),j(t)] \right).
\]

(2.3)

A Fourier transformation with respect to \( x_a \) and \( x_b \) produces the time evolution amplitude in momentum space,

\[
(p_b t_b|p_a t_a)[k(t),j(t)]
\]

\[
= \int_{-\infty}^{\infty} dx_a \int_{-\infty}^{\infty} dx_b \exp \left( -i(p_b x_b - p_a x_a) \right) \times \left( x_b t_b|x_a t_a)[k(t),j(t)] \right).
\]

(2.4)

Here the initial and final momenta \( p_a \) and \( p_b \) are held fixed, so that the right-hand side may be written as the path integral

\[
(p_b t_b|p_a t_a)[k(t),j(t)] = \int_{p_a t_a}^{p_b t_b} Dp(t) Dx(t) \left[ \frac{2\pi\hbar}{i} \right] \exp \left( \frac{i}{\hbar} A[p(t),x(t);k(t),j(t)] \right).
\]

(2.5)

We remark that both path integrals (2.3) and (2.5) are properly defined as continuum limits of ordinary integrals after a time-slicing procedure. Since end points of paths are fixed in coordinate and momentum space, respectively, the discretized expressions for the path integrals turn out to be slightly asymmetric in \( p(t) \) and \( x(t) \) [1], Chap. 2.

The time evolution amplitudes (2.3) and (2.5) with fixed end points can now be reduced to corresponding ones with vanishing end points. For this, we shift the current \( k(t) \) in Eq. (2.1) by a source term \( x_b \delta(t_b - t) - x_a \delta(t - t_a) \) and observe that this produces by Eqs. (2.2) and (2.5) an overall phase factor:

\[
(p_b t_b|p_a t_a)[k(t) + x_b \delta(t_b - t) - x_a \delta(t - t_a),j(t)] \]

\[
= \exp \left( \frac{i}{\hbar} \left( p_b x_b - p_a x_a \right) \right) (p_b t_b|p_a t_a)[k(t),j(t)].
\]

(2.6)

By inverting the Fourier transformation (2.4), the configuration space amplitude (2.3) is seen to satisfy
\( (x_{k|t_0}[x_{t_0} a])[k(t) + x_k \delta(t_0 - t) - x_k \delta(t - t_0), j(t)] \)
\[ = (x_k + x_k t_0[x_k + x_k t_0])[k(t), j(t)], \tag{2.7} \]
where again the \( \delta \) functions are understood as \( \delta(t_0 - \epsilon - t) \) and \( \delta(t - t_0 + \epsilon) \) in the limit \( \epsilon \to 0 \). Because of this relation, the amplitude (2.3) can be reduced to a path integral with vanishing end points but additional \( \delta \) terms in the current \( k(t) \):
\[ (x_{k|t_0}[x_{t_0} a])[k(t), j(t)] \]
\[ = (x_k = 0 t_0[x_k = 0 t_0, k(t) + x_k \delta(t_0 - t) \]
\[ - x_k \delta(t - t_0), j(t)]. \tag{2.8} \]
A similar expression exists, if momentum end points are fixed in momentum space by adding \( p_a \delta(t_0 - t) - p_b \delta(t_0 - t) \) to the current \( j(t) \):
\[ (p_{k|t_0}[p_{t_0} a])[k(t), j(t)] = (p_k = 0 t_0[p_k = 0 t_0, k(t), j(t) \]
\[ + p_a \delta(t_0 - t) - p_b \delta(t_0 - t)]. \tag{2.9} \]

We now explore the consequences of these two relations for the calculation of correlation functions.

### B. Correlation functionals

The functional dependence of the time evolution amplitudes (2.3) and (2.5) on the currents \( k(t) \) and \( j(t) \) allows us to calculate expectation values of arbitrary functionals \( F[p(t), x(t)] \) from the path integral
\[ \langle F[p(t), x(t)][k(t), j(t)] \rangle^{v_{a|t_0}} \]
\[ = \frac{1}{(v_{b|t_0} v_{a|t_0} a) k(t), j(t)] \}
\[ \times \int_{v_{a|t_0}} \frac{Dp(t) Dx(t)}{2 \pi \hbar} \phi[p(t), x(t)] \]
\[ \times \exp \left\{ i \hbar A[p(t), x(t); k(t), j(t)] \right\}, \tag{2.10} \]
where the variable \( v \) may be \( p \) or \( x \). The usual correlation functions
\[ \langle p(t_1) \cdots p(t_n) x(t_{n+1}) \cdots x(t_m) \rangle^{v_{a|t_0}} \]
\[ = \frac{1}{(v_{b|t_0} v_{a|t_0} a) k(t), j(t)] \}
\[ \times \int_{v_{a|t_0}} \frac{Dp(t) Dx(t)}{2 \pi \hbar} p(t_1) \cdots p(t_n) x(t_{n+1}) \cdots x(t_m) \]
\[ \times \exp \left\{ i \hbar A[p(t), x(t); k(t), j(t)] \right\} \tag{2.11} \]
are special cases of Eq. (2.10), so we shall call the general expectation values (2.10) **correlation functionals**. The sources \( k(t) \) and \( j(t) \) permit us to express Eq. (2.10) in terms of functional derivatives:
\[ \langle F[p(t), x(t)][k(t), j(t)] \rangle^{v_{a|t_0}} \]
\[ = \int_{w_b = 0 t_0} \frac{Dp(t) Dx(t)}{2 \pi \hbar} \delta(v(t_n) - v_b) \delta(v(t_b) - v_b) \]
\[ \times \exp \left\{ i \hbar A[p(t), x(t); k(t), j(t)] \right\}, \tag{2.13} \]
where the paths \( v(t) \) stand either for \( p(t) \) or for \( x(t) \). In each of these cases, the paths \( w(t) \) denote the conjugate variables \( x(t) \) or \( p(t) \), respectively. In this form, the path integral possesses the advantage that usual correlation functions (2.11) can be determined by path averages, in which intermediate and end points are treated on equal footing. Indeed, inserting \( \delta \) functions according to
\[ \langle p(t_1) \cdots p(t_n) x(t_{n+1}) \cdots x(t_m) \rangle^{v_{a|t_0}} \]
\[ = \frac{1}{(v_{b|t_0} v_{a|t_0} a) k(t), j(t)] \}
\[ \times \left\{ \int_{w_b = 0 t_0} \frac{Dp(t) Dx(t)}{2 \pi \hbar} \delta(v(t_n) - v_a) \delta(v(t_b) - v_b) \right\} \]
\[ \times \exp \left\{ i \hbar A[p(t), x(t); k(t), j(t)] \right\}, \tag{2.14} \]
we obtain with a similar reasoning
\[ \langle p(t_1) \cdots p(t_n) x(t_{n+1}) \cdots x(t_m) \rangle^{v_{a|t_0}} \]
\[ = \frac{1}{(v_{b|t_0} v_{a|t_0} a) k(t), j(t)] \}
\[ \times \left\{ \int_{w_b = 0 t_0} \frac{Dp(t) Dx(t)}{2 \pi \hbar} \delta(v(t_n) - v_b) \right\} \]
\[ \times \exp \left\{ i \hbar A[p(t), x(t); k(t), j(t)] \right\}. \tag{2.15} \]
C. Standard Hamiltonian

The above formalism can be made more specific for the standard Hamiltonian (1.8). Then the path integrals over the momentum paths \( p(t) \) in Eqs. (2.3) and (2.5) become harmonic and can be explicitly evaluated. The phase-space integral (2.3), for instance, reduces to the configuration space path integral,

\[
(x_{b}|_{a}, x_{a}|_{b})[k(t), j(t)] = \int_{s_{a}}^{s_{b}} Dx(t) \times \exp \left( i \frac{\hbar}{\hbar} A[x(t); k(t), j(t)] \right),
\]

(2.16)

where the current \( k(t) \) couples linearly to the path momentum \( M \dot{x}(t) \) in the action

\[
A[x(t); k(t), j(t)] = \int_{t_{a}}^{t_{b}} dt \left( \frac{M}{2} k^{2}(t) - V(x(t), t) + x(t)j(t) + M \dot{x}(t)(k(t) + \frac{M}{2} \dot{k}^{2}(t)) \right).
\]

(2.17)

A subsequent partial integration transforms the current \( k(t) \) to an effective coordinate current with an extra phase factor:

\[
(x_{b}|_{a}, x_{a}|_{b})[k(t), j(t)] = (x_{b}|_{a}, x_{a}|_{b})[0, j(t) - M k(t)] \times \exp \left( i \frac{\hbar}{\hbar} M \int_{s_{a}}^{s_{b}} dx(k_{b} - k_{a} + \frac{1}{2} \int_{t_{a}}^{t_{b}} dt k^{2}(t)) \right).
\]

(2.18)

Note that combining Eqs. (2.8) and (2.18) proves relation (1.7) for any arbitrary quantum-mechanical system with the standard Hamiltonian (1.8).

In the next section we determine the generating functional \( (x_{b}|_{a}, x_{a}|_{b})[0, j(t)] \) for a harmonic oscillator with arbitrary time-dependent frequency \( \Omega(t) \) and use Eq. (2.18) to construct the full generating functional \( (x_{b}|_{a}, x_{a}|_{b})[k(t), j(t)] \).

III. TIME-DEPENDENT HARMONIC OSCILLATOR

Consider a standard Hamiltonian (1.8) with a harmonic potential containing an arbitrary time-dependent frequency:

\[
V(x, t) = \frac{M}{2} \Omega^{2}(t)x^{2}.
\]

(3.1)

The generating functionals (2.3) and (2.5) are then expressible in terms of two fundamental solutions \( D_{a}(t), D_{b}(t) \) of the corresponding classical equation of motion with particular boundary conditions [2]

\[
\dot{\mathcal{K}}(t)D_{a}(t) = 0, \quad \mathcal{K}(t)D_{b}(t) = 0,
\]

(3.2)

\[
\dot{\mathcal{K}}(t)D_{a}(t) = 0, \quad \mathcal{K}(t)D_{b}(t) = 0,
\]

(3.3)

where \( \dot{\mathcal{K}}(t) \) denotes the operator

\[
\dot{\mathcal{K}}(t) = -\partial_{t}^{2} - \Omega^{2}(t).
\]

(3.4)

Since the time derivative of the Wronski determinant

\[
W(t) = D_{a}(t)D_{b}(t) - D_{a}(t)D_{b}(t)
\]

vanishes, we observe the identity

\[
D_{a}(t_{b}) = D_{b}(t_{a}).
\]

(3.6)

Note that a similar identity does not hold for the time derivatives of the two fundamental solutions \( D_{a}(t) \) and \( D_{b}(t) \). Indeed, partially integrating the differential equation for \( \dot{D}_{a}(t) \) and taking into account Eqs. (3.2)–(3.5), we deduce

\[
\dot{D}_{b}(t_{a}) + \dot{D}_{a}(t_{b}) = -2 \int_{t_{a}}^{t_{b}} dt \Omega(t) \dot{\Omega}(t) D_{a}(t) D_{b}(t).
\]

(3.7)

Let us now determine the time evolution amplitude (2.16) in configuration space for a vanishing current \( k(t) \). We decompose the paths \( x(t) \) into the classical path \( x_{cl}(t) \) and the quantum fluctuations \( \delta x(t) \) around it:

\[
x(t) = x_{cl}(t) + \delta x(t).
\]

(3.8)

The classical path \( x_{cl}(t) \) solves the boundary value problem

\[
\dot{\mathcal{K}}(t)x_{cl}(t) = -\frac{j(t)}{M}, \quad x_{cl}(t_{a}) = x_{a}, \quad x_{cl}(t_{b}) = x_{b},
\]

(3.9)

and the fluctuations \( \delta x(t) \) vanish at the end points:

\[
\delta x(t_{a}) = \delta x(t_{b}) = 0.
\]

(3.10)

Inserting the decomposition (3.8) into the action (2.17), we observe that due to Eqs. (3.9) and (3.10) the total action decomposes into a classical part

\[
\mathcal{A}[x_{cl}(t); 0, j(t)] = \frac{M}{2} \left[ x_{b}x_{cl}^{2}(t_{b}) - x_{a}x_{cl}^{2}(t_{a}) \right] + \frac{1}{2} \int_{t_{a}}^{t_{b}} dt x_{cl}(t)j(t),
\]

(3.11)

and a fluctuation part, which is simply the classical action evaluated for the fluctuations \( \delta x(t) \) at \( j = 0 \):

\[
\mathcal{A}[x(t); 0, j(t)] = \mathcal{A}[x_{cl}(t); 0, j(t)] + \mathcal{A}[\delta x(t); 0, 0],
\]

(3.12)

Inserting this into the original path integral (2.16), it factorizes into the product of a classical amplitude with the classical action (3.11), and an additional fluctuation factor which is equal to the amplitude at vanishing end points:

\[
(x_{b}|_{a}, x_{a}|_{b})[0, j(t)] = \exp \left( i \frac{\hbar}{\hbar} \mathcal{A}[x_{cl}(t); 0, j(t)] \right) \times (x_{b} = 0, t_{b}|_{a} x_{a} = 0, t_{a}|_{b} [0, 0],
\]

(3.13)
A. Classical action

The classical action in the presence of currents can be expressed in terms of the solutions $D_a(t), D_b(t)$ of the time-dependent harmonic boundary value problems (3.2) and (3.3). First we decompose the solution of the boundary value problem (3.9) in the presence of external sources into a homogeneous and an inhomogeneous contribution:

$$x_{cl}(t) = x_{cl}(t) + \Delta x_{cl}(t).$$  \hspace{1cm} (3.14)

The homogeneous solution reads

$$x_{cl}(t) = \frac{D_a(t)x_a + D_b(t)x_b}{D_a(t)},$$  \hspace{1cm} (3.15)

while the inhomogeneous one is given by

$$\Delta x_{cl}(t) = -\frac{1}{M} \int_{t_a}^{t_b} dt' G_{jj}^i(t,t')j(t'),$$  \hspace{1cm} (3.16)

where $G_{jj}^i(t,t')$ denotes the Green function of the classical equation of motion

$$\tilde{K}(t)G_{jj}^i(t,t') = \delta(t-t')$$  \hspace{1cm} (3.17)

with Dirichlet boundary conditions

$$G_{jj}^i(t_a, t') = G_{jj}^i(t_b, t') = 0.$$  \hspace{1cm} (3.18)

From Eq. (3.17) we deduce that the Green function $G_{jj}^i(t,t')$ solves the homogeneous differential equation for $t \neq t'$:

$$\tilde{K}(t)G_{jj}^i(t,t') = 0,$$  \hspace{1cm} (3.19)

and that its first derivative $\partial_t G_{jj}^i(t,t')$ is discontinuous at $t = t'$:

$$\lim_{\epsilon \to 0} [\partial_t G_{jj}^i(t,t')]_{t=t'+\epsilon} - [\partial_t G_{jj}^i(t,t')]_{t=t'-\epsilon} = -1.$$  \hspace{1cm} (3.20)

The Green function itself is continuous around $t = t'$:

$$\lim_{\epsilon \to 0} [G_{jj}^i(t,t')]_{t=t'+\epsilon} - [G_{jj}^i(t,t')]_{t=t'-\epsilon} = 0.$$  \hspace{1cm} (3.21)

The solution of Eqs. (3.18)–(3.21) is given by Wronski’s famous expression

$$G_{jj}^i(t,t') = \frac{\Theta(t-t')D_b(t)D_a'(t') + \Theta(t'-t)D_a(t)D_b'(t')}{D_a(t_b)},$$  \hspace{1cm} (3.22)

where $\Theta(t-t')$ denotes the Heaviside function which vanishes for $t < t'$ and is equal to unity for $t > t'$. Inserting Eqs. (3.14) and (3.16) we obtain for the classical action (3.11)

$$A[x_{cl}(t); 0, j(t)] = \frac{M}{2D_a(t_b)} \left[ D_a(t_b)x_a^2 - D_b(t_a)x_a^2 - 2x_a x_b \right]$$

$$+ \int_{t_a}^{t_b} dt \ x_{cl}(t) j(t),$$

$$- \frac{1}{2M} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' G_{jj}^i(t,t')j(t)j(t'),$$  \hspace{1cm} (3.23)

where $x_{cl}(t)$ and $G_{jj}^i(t,t')$ are given by Eqs. (3.15) and (3.22), respectively.

B. Fluctuation factor

Now we calculate the fluctuation factor in Eq. (3.13). Recalling the path representation (2.16) with the action (2.17), we have to evaluate

$$(x_b = 0 | x_a = 0, j_a)[0,0] = \int_{\delta x_a = 0, \delta x_b = 0} D\delta x(t) \exp \left[ \frac{iM}{2\hbar} \int_{t_a}^{t_b} dt \ \delta x(t) \tilde{K}(t) \delta x(t) \right].$$  \hspace{1cm} (3.24)

To this end we decompose the fluctuation $\delta x(t)$ in Eq. (3.24) into eigenfunctions $x_n(t)$ of the operator $\tilde{K}(t)$ in Eq. (3.4) with Dirichlet boundary conditions

$$\tilde{K}(t)x_n(t) = \lambda_n x_n(t); \quad x_n(t_a) = x_n(t_b) = 0$$  \hspace{1cm} (3.25)

which satisfy the orthonormality and completeness relations

$$\int_{t_a}^{t_b} dt \ x_n(t) x_n(t') = \delta(t-t'),$$  \hspace{1cm} (3.26)

$$\sum_n x_n(t) x_n(t') = \delta(t-t'),$$  \hspace{1cm} (3.27)

as follows:

$$\delta x(t) = \sum_n c_n x_n(t).$$  \hspace{1cm} (3.28)

The path integral over all possible fluctuations $\delta x(t)$ in Eq. (3.24) amounts to a product of integrals over all expansion coefficients $c_n$:

$$\int_{\delta x_a = 0, \delta x_b = 0} D\delta x(t) = \prod_n \int_{-\infty}^{+\infty} dc_n.$$  \hspace{1cm} (3.29)

The Jacobian determinant $J$ of the transformation (3.28) is an irrelevant constant. Applying Eqs. (3.25)–(3.29), the path integral (3.24) is finally determined by

$$(x_b = 0 | x_a = 0, j_a)[0,0] = \frac{J}{\sqrt{\det \tilde{K}(t)}}.$$  \hspace{1cm} (3.30)

where the determinant of the operator $\tilde{K}(t)$ is equal to the product of its eigenvalues.
\[ \det \hat{K}(t) = \prod_n \lambda_n. \]  

\[ (x_b = 0, t_b | x_a = 0, t_a)[0,0] \]

\[ = \left( \frac{D^{2}_{a}((t_b))}{D^{2}_{a}(t_b)} \right)^{1/2} \]  

\[ (x_b = 0, t_b | x_a = 0, t_a)[0,0] \]

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]

\[ = \left( \frac{M}{2 \pi i \hbar} \right)^{1/2} \]  

\[ \times \exp \left[ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right] \]
and $p_{cl}(t)$ denotes the classical momentum $p_{cl}(t) = M\dot{x}_{cl}(t)$. The Green function $G^i_{jk}(t,t')$ is given by Eq. (3.22), while the others are

$$G^i_{jk}(t,t') = \Theta(t-t')D_{a}(t)D_{a}(t') + \Theta(t-t')\dot{D}_{a}(t')D_{b}(t')$$

$$= G^i_{kj}(t',t),$$

$$G^i_{kk}(t,t') = \Theta(t-t')\dot{D}_{b}(t)\dot{D}_{a}(t') + \Theta(t-t')\dot{D}_{a}(t')\dot{D}_{a}(t')$$

$$= G^i_{kk}(t',t).$$

By differentiating Eq. (3.43) functionally with respect to $j$ and $k$, we see that the Green functions correspond to the correlation functions

$$\langle \bar{x}(t)\bar{x}(t')\rangle[0,0]_{\bar{x}_{a}\bar{x}_{a}} = \frac{i\hbar}{M}G^1_{jk}(t,t'),$$

$$\langle \bar{x}(t)\bar{p}(t')\rangle = \langle [0,0]_{\bar{x}_{a}\bar{x}_{a}}\rangle = ihG^2_{jk}(t,t') = ihG^2_{kj}(t',t),$$

$$\langle \bar{p}(t)\bar{p}(t')\rangle[0,0]_{\bar{x}_{a}\bar{x}_{a}} = ihMG^1_{kk}(t,t').$$

with $\bar{x}(t) = x(t) - x_{cl}(t)$ and $\bar{p}(t) = p(t) - p_{cl}(t)$. These results can be summarized by the mnemonic rule that the Green functions involving a momentum current $k(t)$ once or twice follow from $G^i_{jk}(t,t')$ by one or two time derivatives if the time derivatives of the Heaviside functions are neglected:

$$G^i_{jk}(t,t') = \frac{n\partial G^i_{jk}(t,t')}{\partial t}t^n, \quad G^i_{kj}(t',t) = \frac{n\partial G^i_{kj}(t,t')}{\partial t}t^n,$$

$$G^i_{kk}(t,t') = \frac{n\partial^2 G^i_{kk}(t,t')}{{\partial t}^2}t^n.$$

A complete analogous expression to Eq. (3.43) is found for the time evolution amplitude in the momentum representation. The Fourier transformation (2.4) of Eq. (3.42) yields a Van Vleck–Pauli Morette type of formula

$$(p_{a}p_{a}[k(t),j(t)])$$

$$= \left(2\pi\hbar\right)^{1/2} \frac{\partial^2 A(p_{b},t_{b};p_{a},t_{a})[k(t),j(t)]}{\partial p_{a}\partial p_{a}} \exp\left\{ \frac{i}{\hbar}A(p_{b},t_{b};p_{a},t_{a})[k(t),j(t)]\right\} ,$$

where the action is the Legendre transform of Eq. (3.43),

$$A(p_{b},t_{b};p_{a},t_{a})[k(t),j(t)] = A(x_{b},t_{b};x_{a},t_{a})[k(t),j(t)] - p_{b}x_{b} + p_{a}x_{a},$$

calculated for the conjugate variables

$$p_{b} = \frac{\partial A(x_{b},t_{b};x_{a},t_{a})[k(t),j(t)]}{\partial x_{b}},$$

$$p_{a} = \frac{\partial A(x_{b},t_{b};x_{a},t_{a})[k(t),j(t)]}{\partial x_{a}}.$$

This brings Eq. (3.51) to the form

$$A(p_{b},t_{b};p_{a},t_{a})[k(t),j(t)]$$

$$= \frac{D_{a}(t_{a})\left[D_{a}(t)\bar{D}_{a}(t_{a}) + p_{a}\bar{D}_{a}(t_{a}) - D_{b}(t_{a})\right]}{M[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]}$$

$$+ \int_{t_{a}}^{t_{b}}dt\int_{t_{a}}^{t_{b}}dt'\left[\frac{1}{M}G^1_{jj}(t,t')j(t')j(t)
+ G^2_{jk}(t,t')j(t')k(t')
+ MG^1_{kk}(t,t')k(t')k(t')\right],$$

where the classical solution now reads

$$\bar{x}_{cl}(t) = p_{a}\left[D_{a}(t) + D_{a}(t)\bar{D}_{a}(t_{a}) + p_{a}\bar{D}_{a}(t_{a}) - D_{b}(t_{a})\right]$$

$$M[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})],$$

and $\bar{p}_{cl}(t)$ denotes the associated classical momentum $\bar{p}_{cl}(t) = M\dot{x}_{cl}(t)$. The Green functions in Eq. (3.53) turn out to be

$$G^1_{jj}(t,t') = \Theta(t-t')\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t)\bar{D}_{a}(t_{a}) - D_{b}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]}$$

$$+ \Theta(t'-t)\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]} = G^1_{jj}(t,t'),$$

$$G^2_{jk}(t,t') = \Theta(t-t')\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]}$$

$$+ \Theta(t'-t)\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]} = G^2_{jk}(t,t'),$$

$$G^1_{kk}(t,t') = \Theta(t-t')\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]}$$

$$+ \Theta(t'-t)\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]} = G^1_{kk}(t,t'),$$

$$G^2_{kk}(t,t') = \Theta(t-t')\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]}$$

$$+ \Theta(t'-t)\frac{[D_{a}(t)\bar{D}_{a}(t_{a}) - D_{a}(t)\bar{D}_{a}(t_{a}) + D_{a}(t_{a})]}{D_{a}(t_{a})[1 + \bar{D}_{a}(t_{a})\bar{D}_{a}(t_{a})]} = G^2_{kk}(t,t').$$
\[ G^p_p(t,t') = \Theta(t-t') \frac{[D_b(t)\hat{D}_a(t_b)+D_a(t)][\hat{D}_a(t')\hat{D}_b(t_a)-\hat{D}_b(t')]}{D_a(t_b)[1+D_a(t_b)D_b(t_a)]} + \Theta(t'-t) \frac{[D_a(t)\hat{D}_b(t_a)-D_b(t)][\hat{D}_b(t')\hat{D}_a(t_b)+\hat{D}_a(t')]}{D_a(t_b)[1+D_a(t_b)D_b(t_a)]} = G^p_p(t',t), \]  
\[ G^p_{hk}(t,t') = \Theta(t-t') \frac{[D_b(t)\hat{D}_a(t_h)+D_a(t)][D_a(t')\hat{D}_b(t_k)-D_b(t')]}{D_a(t_h)[1+D_a(t_h)D_b(t_k)]} + \Theta(t'-t) \frac{[D_a(t)\hat{D}_b(t_k)-D_b(t)][D_a(t')\hat{D}_b(t_h)+D_b(t')]}{D_a(t_h)[1+D_a(t_h)D_b(t_k)]} = G^p_{hk}(t',t). \]

The relation to the correlation functions is similar to Eqs. (3.46)–(3.48): 
\[ \langle \tilde{x}(t)\tilde{x}(t') \rangle[0,0] = \frac{i\hbar}{M} G^a_a(t,t'), \]
\[ \langle \tilde{p}(t)\tilde{p}(t') \rangle[0,0] = i\hbar G^b_b(t,t') = i\hbar G^\phi^{(1)}_{b}(t,t'), \]
\[ \langle \tilde{p}(t)\tilde{p}(t') \rangle[0,0] = i\hbar M G^a_a(t,t'), \]

with \( \tilde{x}(t) = x(t) - x_a(t) \) and \( \tilde{p}(t) = p(t) - \bar{p}_a(t) \). The relation between the similar-looking actions (3.43) and (3.53) becomes more transparent by reexpressing both in terms of partial derivatives of the classical solutions \( x_a(t), x_b(t), \bar{p}_a(t), \bar{p}_b(t) \) with respect to the end points \( x_b, x_a, \bar{p}_b, \bar{p}_a \), respectively. In the configuration representation we obtain 
\[ A(x_b,t_b;x_a,t_a)[k(t),j(t)] = \frac{1}{2} \left( x_b, x_a \right) \left( \frac{\partial p_b}{\partial x_b} \frac{\partial p_b}{\partial x_a} \right) x_b x_a + \int_{t_a}^{t_b} dt x_b x_a \left( \frac{\partial p_a(t)}{\partial x_b} \frac{\partial p_a(t)}{\partial x_a} \right) \frac{\partial k(t)}{\partial x_b} \frac{\partial k(t)}{\partial x_a} 
\[ - \frac{1}{2} \left( \frac{\partial x_b}{\partial p_a} \right) \left[ \int_{t_a}^{t_b} dt' \left( \frac{\partial k(t')}{\partial x_b} \frac{\partial k(t')}{\partial x_a} \right) \right] 
\[ + \Theta(t'-t) \left( \frac{\partial p_a(t)}{\partial x_b} \frac{\partial p_a(t)}{\partial x_a} \right) \left[ \frac{\partial x_b(t)}{\partial x_b} \frac{\partial x_b(t)}{\partial x_a} \frac{\partial x_a(t)}{\partial x_b} \frac{\partial x_a(t)}{\partial x_a} \right] \left[ \frac{\partial k(t)}{\partial x_b} \frac{\partial k(t)}{\partial x_a} \right]. \]

The momentum representation, on the other hand, has the analogous form with \( x \) and \( p \) interchanged:
\[ A(p_b,t_b;p_a,t_a)[k(t),j(t)] = \frac{1}{2} \left( p_b, p_a \right) \left( \frac{\partial x_b}{\partial p_b} \frac{\partial x_b}{\partial p_a} \right) p_b p_a + \int_{t_a}^{t_b} dt \left( p_b, p_a \right) \left( \frac{\partial \bar{p}_a(t)}{\partial p_b} \frac{\partial \bar{p}_a(t)}{\partial p_a} \right) \frac{\partial k(t)}{\partial p_b} \frac{\partial k(t)}{\partial p_a} 
\[ - \frac{1}{2} \left( \frac{\partial x_b}{\partial p_a} \right) \left[ \int_{t_a}^{t_b} dt' \left( \frac{\partial k(t')}{\partial p_b} \frac{\partial k(t')}{\partial p_a} \right) \right] 
\[ + \Theta(t'-t) \left( \frac{\partial \bar{p}_a(t)}{\partial p_b} \frac{\partial \bar{p}_a(t)}{\partial p_a} \right) \left[ \frac{\partial x_b(t)}{\partial p_b} \frac{\partial x_b(t)}{\partial p_a} \frac{\partial x_a(t)}{\partial p_b} \frac{\partial x_a(t)}{\partial p_a} \right] \left[ \frac{\partial k(t)}{\partial p_b} \frac{\partial k(t)}{\partial p_a} \right]. \]

These expressions for the generating functionals (3.42) and (3.50) exhibit clearly the symmetry properties (2.8) and (2.9).
IV. SMEARING FORMULA FOR HARMONIC FLUCTUATIONS

As a first application of the generating functional (3.42) we derive a general rule for calculating correlation functions of polynomial or nonpolynomial functions of \(x(t)\) and \(p(t)\). The result will be expressed in the form of a "smearing formula." This formula will represent an essential tool for calculating perturbation expansions with nonpolynomial interactions. Such expansions serve in variational perturbation theory to obtain convergent approximations for quantum-statistical partition functions [6] or density matrices [7].

Consider the correlation functions of a product of local functions for vanishing currents, 

\[
\langle F_1(x(t_1))F_2(x(t_2))\cdots F_N(x(t_N))F_{N+1}(p(t_{N+1}))F_{N+2}(p(t_{N+2}))\cdots F_{N+M}(p(t_{N+M})) \rangle^{x_0}_{t_0} 
\]

where the harmonic time evolution amplitude with zero external currents (transforming the functions into covariances serve in variational perturbation theory to obtain convergent approximations for quantum-statistical partition functions [6] or density matrices [7].

Inserting these equations into the action (3.42), the correlation functions (4.1) may be reexpressed as

\[
\langle F_1(x(t_1))\cdots F_{N+M}(p(t_{N+M})) \rangle^{x_0}_{t_0} = \frac{1}{(x_0|p_0|a_0)} \prod_{n=1}^{N} \left[ \int_{-\infty}^{+\infty} dx_n F_n(x_n) \int_{-\infty}^{+\infty} \frac{d\xi_n}{2\pi} e^{i\xi_n x_n} \right] \prod_{m=1}^{M} \left[ \int_{-\infty}^{+\infty} dp_m F_{N+m}(p_m) \int_{-\infty}^{+\infty} d\kappa_m e^{-i\kappa_m[p_m-p(t_{N+m})]/\hbar} \right] 
\]

where the generating functional is given by Eq. (3.42). The currents \(j(t)\) and \(k(t)\) are specialized to

\[
j(t) = -\hbar \sum_{n=1}^{N} \xi_n \delta(t-t_n), \quad k(t) = \sum_{m=1}^{M} \kappa_m \delta(t-t_{N+m}).
\]

Inserting these equations into the action (3.43) and the Green functions (3.32), (3.44), and (3.45), we find the Fourier decomposition of the generating functional (3.42), so that the correlation functions (4.4) become

\[
\langle F_1(x(t_1))\cdots F_{N+M}(p(t_{N+M})) \rangle^{x_0}_{t_0} = \prod_{n=1}^{N} \left[ \int_{-\infty}^{+\infty} dx_n F_n(x_n) \int_{-\infty}^{+\infty} \frac{d\xi_n}{2\pi} e^{i\xi_n x_n} \right] \prod_{m=1}^{M} \left[ \int_{-\infty}^{+\infty} dp_m F_{N+m}(p_m) \int_{-\infty}^{+\infty} d\kappa_m e^{-i\kappa_m[p_m-p(t_{N+m})]/\hbar} \right] 
\]

where we used the abbreviations

\[
G_{m}^{n'} = G_{m}^{n}(t_n,t_n'), \quad G_{k}^{m} = G_{k}(t_n,t_{N+m}), \quad G_{kk}^{mm'} = G_{kk}(t_{N+m},t_{N+m'}).
\]

To proceed, it is more convenient to write expression (4.6) as a convolution integral

\[
\langle F_1(x(t_1))\cdots F_{N+M}(p(t_{N+M})) \rangle^{x_0}_{t_0} = \prod_{n=1}^{N} \left[ \int_{-\infty}^{+\infty} dx_n F_n(x_n) \int_{-\infty}^{+\infty} \frac{d\xi_n}{2\pi} e^{i\xi_n x_n} \right] \prod_{m=1}^{M} \left[ \int_{-\infty}^{+\infty} dp_m F_{N+m}(p_m) \int_{-\infty}^{+\infty} d\kappa_m e^{-i\kappa_m[p_m-p(t_{N+m})]/\hbar} \right] 
\]

where we used the abbreviations

\[
G_{m}^{n'} = G_{m}^{n}(t_n,t_n'), \quad G_{k}^{m} = G_{k}(t_n,t_{N+m}), \quad G_{kk}^{mm'} = G_{kk}(t_{N+m},t_{N+m'}).
\]
involving the Gaussian distribution
\[ P(x_1, \ldots, p_M) = \frac{1}{(2\pi)^N} \int d^{N+M}v \exp \left[ i \mathbf{w}^T \mathbf{v} - \frac{i}{2} \mathbf{v}^T G \mathbf{v} \right]. \]  
(4.9)

The dimensionless vectors \( \mathbf{v} \) and \( \mathbf{w} \) have \( N+M \) components and are defined as
\[ \mathbf{v}^T = \left( \frac{\hbar}{M \Omega} \right)^{1/2} \xi_1, \ldots, \left( \frac{\hbar}{M \Omega} \right)^{1/2} \xi_N, \left( \frac{\hbar}{M \Omega} \right)^{1/2} \kappa_1, \ldots, \left( \frac{\hbar}{M \Omega} \right)^{1/2} \kappa_M \]
(4.10)
and
\[ \mathbf{w}^T = \left( \frac{M \Omega}{\hbar} \right)^{1/2} [x_{11} - x_{10}(t_1)], \ldots, \left( \frac{M \Omega}{\hbar} \right)^{1/2} [x_N - x_{0N}(t_N)], -\frac{1}{\hbar M \Omega} [p_{11} - p_{01}(t_{11})], \ldots, -\frac{1}{\hbar M \Omega} [p_{M1} - p_{0M}(t_{N+M})]. \]
(4.11)

The \((N+M) \times (N+M)\) matrix of Green functions
\[ G = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \]
(4.12)
can be decomposed into block matrices \( A \), \( B \), and \( C \). The \( N \times N \) matrix \( A \) and the \( M \times M \) matrix \( C \) are defined by
\[ A = \Omega \begin{pmatrix} G_{11}^{11} & G_{11}^{12} & \cdots & G_{11}^{1N} \\ G_{12}^{11} & G_{12}^{12} & \cdots & G_{12}^{1N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{1N}^{11} & G_{1N}^{12} & \cdots & G_{1N}^{1N} \end{pmatrix}, \]
\[ C = \frac{1}{\Omega} \begin{pmatrix} G_{kk}^{1k} & G_{kk}^{12} & \cdots & G_{kk}^{1M} \\ G_{kk}^{2k} & G_{kk}^{22} & \cdots & G_{kk}^{2M} \\ \vdots & \vdots & \ddots & \vdots \\ G_{kk}^{kM} & G_{kk}^{2M} & \cdots & G_{kk}^{11} \end{pmatrix} \]
(4.13)
and yield quadratic forms of the position and momentum variables, respectively. The \( N \times M \) matrix
\[ B = \begin{pmatrix} -G_{jj}^{11} & -G_{jk}^{12} & \cdots & -G_{jk}^{1M} \\ -G_{jk}^{21} & -G_{jk}^{11} & \cdots & -G_{jk}^{2M} \\ \vdots & \vdots & \ddots & \vdots \\ -G_{jk}^{N1} & -G_{jk}^{N2} & \cdots & -G_{jk}^{NM} \end{pmatrix} \]
(4.14)
gives rise to quadratic terms which are linear in both position and momentum variables. The multidimensional integral in Eq. (4.9) is of the Fresnel type and can easily be done, yielding an explicit expression for the Gaussian distribution (4.9), where \( G^{-1} \) represents the matrix inverse of Eq. (4.12) whose block form is
\[ G^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}BC^{-1} \\ -C^{-1}B^TX^{-1} & C^{-1}+C^{-1}B^TX^{-1}BC^{-1} \end{pmatrix} \]
(4.16)
with the abbreviation
\[ X = A - BC^{-1}B^T. \]
(4.17)
Since the matrix \( G \) may be decomposed as
\[ G = \begin{pmatrix} 1 & B \\ 0 & C \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & C^{-1}B^T \end{pmatrix}, \]
(4.18)
when the matrix \( C \) is regular, the determinant of \( G \) factorizes as follows:
\[ \det G = \det C \det X. \]
(4.19)
For singular matrix \( C \) but \( A \) regular, one may make use of another decomposition,
\[ G = \begin{pmatrix} 1 & B \\ B^TA^{-1} & X' \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}, \]
(4.20)
with \( X' = C - B^TA^{-1}B \). Then the determinant of \( G \) is given by
\[ \det G = \det X' \det A. \]
(4.21)
With the Gaussian distribution (4.15), our result (4.8) constitutes a smearing formula which describes the effect of harmonic fluctuations upon arbitrary products of functions of space and momentum variables at different times.

V. GENERALIZED WICK RULES AND FEYNMAN DIAGRAMS

In applications, there often occur correlation functions for mixtures of nonpolynomial functions $F(\vec{x}(t))$ or $F(\vec{p}(t))$ and powers according to this section generalized Wick rules and Feynman diagrams. In order to evaluate such correlation functions, we derive in successively applying the derivative rule

$$G^{(n)}(t_1, \ldots, t_n) = \langle \vec{x}(t_1) \cdots \vec{x}(t_n) \rangle_{\Omega}^{t_1 \cdots t_n}.$$  

Note that it will be sufficient to study only the correlation functions involving the deviations from the classical path, respectively. This expectation value can be decomposed with the help of Wick’s expansion,

$$G^{(n)}(t_1, \ldots, t_n) = \sum_{\text{pairs}} G^{(2)}(t_{p(1)}, t_{p(2)}) \cdots G^{(2)}(t_{p(n-1)}, t_{p(n)}),$$

where $\rho$ denotes the operation of pairwise index permutation. Thereby, the Green function $G^{(2)}(t_1, t_2)$ is already given by Eq. (3.46). Note that Eq. (5.1) may be considered as a consequence of a simple derivative rule

$$\langle F(\vec{x}(t_1))\vec{x}(t_2) \rangle_{\Omega}^{t_1 \cdots t_2} = \langle \vec{x}(t_1)\vec{x}(t_2) \rangle_{\Omega}^{t_1 \cdots t_2} \langle F^*(\vec{x}(t_1)) \rangle_{\Omega}^{t_1 \cdots t_2}$$

with $F^*(\vec{x}) = \partial F(\vec{x})/\partial \vec{x}$. By applying this recursively, one eventually obtains Eq. (5.3). And conversely, the derivative rule (5.4) can be proved for polynomial functions $F(\vec{x}(t))$, following directly from Wick’s theorem (5.3).

The two-point Green functions $G^{(2)}(t_1, t_2)$, occurring in Eq. (5.3), can be considered as a Wick contraction, which we introduce as follows:

$$\langle F(\vec{x}(t_1))\vec{x}(t_2) \rangle_{\Omega}^{t_1 \cdots t_2} = \langle \vec{x}(t_1)\vec{x}(t_2) \rangle_{\Omega}^{t_1 \cdots t_2} \langle F^*(\vec{x}(t_1)) \rangle_{\Omega}^{t_1 \cdots t_2}$$

with the multiplicity factor.

Decomposing polynomial correlations of $\vec{x}(t)$ and $\vec{p}(t)$ with the help of these contractions corresponding to Eq. (5.3) or successively applying the derivative rule (5.4) leads to the following results:

$$\langle \vec{x}(t_1)\vec{x}(t_2) \rangle_{\Omega}^{t_1 \cdots t_2} = \sum_{l=\alpha, \alpha + 2, \alpha + 4, \ldots}^{\text{max}_{n, m}} \sum_{l=\alpha, \alpha + 2, \alpha + 4, \ldots}^{\text{max}_{n, m}} c_l \left[ i \hbar M G_{jj}(t_1, t_2) \right]^{(n-1)/2} \left[ i \hbar M G_{jj}(t_1, t_2) \right]^{(m-1)/2},$$

$$\langle \vec{p}(t_1)\vec{p}(t_2) \rangle_{\Omega}^{t_1 \cdots t_2} = \sum_{l=\alpha, \alpha + 2, \alpha + 4, \ldots}^{\text{max}_{n, m}} \sum_{l=\alpha, \alpha + 2, \alpha + 4, \ldots}^{\text{max}_{n, m}} c_l \left[ i \hbar M G_{kk}(t_1, t_2) \right]^{(n-1)/2} \left[ i \hbar M G_{kk}(t_1, t_2) \right]^{(m-1)/2}.$$
Note that $(−k)!! = 1$ for any positive integer $k$. For nonvanishing correlation, the sum $n+m$ must be even so that the regulation parameter $α$ is defined as follows:

$$α = \begin{cases} 0, & n,m \text{ even}, \\ 1, & n,m \text{ odd}. \end{cases}$$  (5.14)

The contractions defined in Eqs. (5.5)–(5.8) can be used to treat Taylor-expandable functions $F(\bar{x}(t))$ and $F(\bar{p}(t))$ only. The desired derivative rules for such correlations read

$$\left\langle F(\bar{x}(t)) \right\rangle^{x_p,x_a}_\Omega = \sum_{l = a,a+2,a+4,\ldots}^{n} \frac{n!}{(n-l)!!!(n-l)!} \left[ \frac{i\hbar}{M} G_{jj}(t_2,t_2) \right]^{(n-l)/2} \left[ \frac{i\hbar}{M} G_{jk}(t_1,t_2) \right]^{l} \left\langle F^{(l)}(\bar{x}(t_1)) \right\rangle^{x_p,x_a}_\Omega, \quad (5.15)$$

$$\left\langle F(\bar{p}(t_1)) \right\rangle^{p_p,x_a}_\Omega = \sum_{l = a,a+2,a+4,\ldots}^{n} \frac{n!}{(n-l)!!!(n-l)!} \left[ i\hbar MG_{kk}(t_2,t_2) \right]^{(n-l)/2} \left[ i\hbar MG_{kk}(t_1,t_2) \right]^{l} \left\langle F^{(l)}(\bar{p}(t_1)) \right\rangle^{x_p,x_a}_\Omega, \quad (5.16)$$

$$\left\langle F(\bar{p}(t_1)) \right\rangle^{p_p,x_a}_\Omega = \sum_{l = a,a+2,a+4,\ldots}^{n} \frac{n!}{(n-l)!!!(n-l)!} \left[ i\hbar MG_{kk}(t_2,t_2) \right]^{(n-l)/2} \left[ i\hbar MG_{kk}(t_1,t_2) \right]^{l} \left\langle F^{(l)}(\bar{p}(t_1)) \right\rangle^{x_p,x_a}_\Omega, \quad (5.17)$$

$$\left\langle F(\bar{p}(t_1)) \right\rangle^{p_p,x_a}_\Omega = \sum_{l = a,a+2,a+4,\ldots}^{n} \frac{n!}{(n-l)!!!(n-l)!} \left[ \frac{i\hbar}{M} G_{jj}(t_2,t_2) \right]^{(n-l)/2} \left[ i\hbar MG_{kk}(t_1,t_2) \right]^{l} \left\langle F^{(l)}(\bar{p}(t_1)) \right\rangle^{x_p,x_a}_\Omega. \quad (5.18)$$

The parameter $α$ distinguishes between even and odd power $n$:

$$α = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases}$$  (5.19)

since even (odd) powers of $n$ lead to even (odd) derivatives of the function $F(\bar{x}(t_1))$. The $l$th derivative $F^{(l)}(\bar{x}(t_1))$ is formed with respect to $x(t_1)$, and $F^{(l)}(\bar{p}(t_1))$ is the $l$th derivative with respect to $p(t_1)$. Note that in the last line the Green function $G_{jk}$ appears with exchanged time arguments, which in this case happens to be inessential due to the symmetry $G_{jk}(t_2,t_1) = G_{kj}(t_1,t_2)$.

B. Generalized Wick rule

According to their derivation, the contractions (5.15)–(5.18) are only applicable to functions $F(\bar{x}(t))$ and $F(\bar{p}(t))$ which can be Taylor-expanded. In the following, we will show with the help of the smearing formula (4.8) that these derivative rules remain valid for functions $F(\bar{x}(t))$ and $F(\bar{p}(t))$ with Laurent expansions. Expectations of this type appear in variational perturbation theory (see for position-position coupling Ref. [7]). Since the proceeding is similar in all the cases (5.15)–(5.18), we shall only discuss the expectation value

$$\left\langle F(\bar{x}(t_1)) \right\rangle^{x_p,x_a}_\Omega$$  (5.20)

in detail. For this we consider the generating functional of all such expectation values following from Eq. (4.8).

$$\left\langle F(\bar{x}(t_1)) e^{i\bar{p}(t_2)} \right\rangle^{x_p,x_a}_\Omega = \frac{1}{\sqrt{-\det G}} \frac{d}{dx} \left[ \frac{1}{2\pi\hbar} \right]^{+\infty}_{-\infty} dp \exp \left\{ \frac{i}{\hbar} \left[ \frac{M}{\hbar} G_{kk}(t_2,t_2) x^2 - 2 \frac{1}{\hbar} G_{jk}(t_1,t_2) xp + \frac{1}{\hbar M} G_{jj}(t_1,t_1) p^2 \right] \right\}. \quad (5.21)$$

The $p$ integration can easily be done, leading to
\[
\langle F(\tilde{x}(t_1)) e^{i p\tilde{x}(t_2)} \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a} = e^{i\hbar MG_{l_1}(t_1,t_2)} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi i\hbar G_{l_1}(t_1,t_2) / M}} F[x + i\hbar G_{l_1}(t_1,t_2)] e^{i(Mx^2 / 2\hbar G_{l_1}(t_1,t_2))} \\
= e^{i\hbar MG_{l_1}(t_1,t_2)} \sum_{l=0}^{\infty} \frac{1}{l!} (i\hbar G_{l_1}(t_1,t_2)) l \langle F^{(l)}(\tilde{x}(t_1)) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a}.
\] (5.22)

The correlation of two functions at different times has been reduced to a single-time expectation value of the \(l\)th derivative of the function \(F(\tilde{x}(t_1))\) with respect to \(x(t_1)\), denoted by \(F^{(l)}(\tilde{x}(t_1))\), with Green functions describing the dependence on the second time. Expanding both sides in powers of \(j\), we reobtain Eq. (5.16).

Now we demonstrate that the derivative rules (5.15)–(5.18) for Laurent-expandable functions \(F(\tilde{x}(t))\) and \(F(\tilde{p}(t))\) also follow from generalized Wick rules. Without restriction of universality, we only consider the expectation value

\[
\langle F(\tilde{x}(t_1)) \tilde{x}^n(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a}.
\] (5.23)

The proceeding to reduce the power of the polynomial at the expense of the function \(F(\tilde{x}(t_1))\) is as follows.

(1a) If possible \((n \geq 2)\), contract \(\tilde{x}(t_1)\tilde{x}(t_2)\) with multiplicity \((n-1)\), giving

\[
\langle F(\tilde{x}(t_1)) \tilde{x}^n(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a} = (n-1) \tilde{x}(t_2) \tilde{x}(t_2) \langle F(\tilde{x}(t_1)) \tilde{x}^{n-2}(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a}
\]

with the contraction rules defined in Eqs. (5.5) and (5.25). For \(n=1\), we obtain

\[
\langle F(\tilde{x}(t_1)) \tilde{x}(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a} = \tilde{x}(t_1) \tilde{x}(t_2) \langle F'(x(t_1)) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a},
\] (5.27)

which is valid for any function \(F(\tilde{x}(t))\) generalizing the rule (5.4) that was proved for polynomial functions only. Recursively applying this power reduction, we finally end up with the derivative rule (5.15). Note that the generalization of Wick's rule for mixed position momentum or pure momentum couplings is done along similar lines, leading to the derivative rules (5.16)–(5.18).

C. New Feynman-like rules for nonpolynomial interactions

Higher-order perturbation expressions become usually complicated. For simple polynomial interactions, Feynman diagrams are a useful tool to classify perturbative contributions with the help of graphical rules. Here, we are going to set up analogous diagrammatic rules for perturbation expansions for nonpolynomial interactions \(V(x(t), p(t))\), whose contributions may be expressed as expectations values

\[
\int_{t_0}^{t_0} dt_{\text{a}} \cdots dt_{\text{a}} \langle V(x(t_0), p(t_0)) \cdots V(x(t_1), p(t_1)) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a}.
\] (5.28)

From Eqs. (5.5)–(5.8) it follows that we have four basic propagators whose graphical representation may be defined as (setting \(\hbar = M = 1\) from now on)

\[
\begin{align*}
G_{\text{j}}(t_1, t_2) &\equiv \langle \tilde{x}(t_1) \tilde{x}(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a} = iG_{\text{j}}(t_1, t_2), \\
G_{\text{k}}(t_1, t_2) &\equiv \langle \tilde{p}(t_1) \tilde{p}(t_2) \rangle_{\Omega}^{\text{x}_a} = iG_{\text{k}}(t_1, t_2), \\
G_{\text{jk}}(t_1, t_2) &\equiv \langle \tilde{x}(t_1) \tilde{p}(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a} = iG_{\text{jk}}(t_1, t_2), \\
G_{\text{kj}}(t_1, t_2) &\equiv \langle \tilde{p}(t_1) \tilde{x}(t_2) \rangle_{\Omega}^{T_{\text{b}} - \text{x}_a} = iG_{\text{kj}}(t_1, t_2),
\end{align*}
\]

A vertex is represented as usual by a small dot. The time variable is integrated over at a vertex in a perturbation expansion,

\[
\bullet = \int_{t_0}^{t_0} dt.
\]

We now introduce the diagrammatic representations of the expectation value of arbitrary functions \(F(\tilde{x}(t))\) or \(F(\tilde{p}(t))\) and their derivatives as
With these elements, we can compose Feynman graphs for two-point correlation functions of the type (5.1) for arbitrary \( n \) by successively applying the generalized Wick rule (5.26) or directly using the derivative relations (5.15)–(5.18). The general results become obvious by giving explicitly a graphical representation of the following four correlation functions:

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{\omega}(t_1))\bar{\omega}(t_2) \rangle_{\Omega}^{t_b-t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \ iG_{jj}(t_1,t_2) \langle (F'\bar{\omega}(t_1)) \rangle_{\Omega}^{t_b-t_a}
\]

\[\equiv \star \quad \Rightarrow \quad (5.29)\]

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{\omega}(t_1))\bar{\omega}(t_2) \rangle_{\Omega}^{t_b-t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \ iG_{jj}(t_1,t_2) \langle F'(\bar{\omega}(t_1)) \rangle_{\Omega}^{t_b-t_a} 
\]

\[\equiv \star \quad \Rightarrow \quad (5.30)\]

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{\omega}(t_1))\bar{\omega}(t_2) \rangle_{\Omega}^{t_b-t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \ iG_{jj}(t_1,t_2) \langle F''(\bar{\omega}(t_1)) \rangle_{\Omega}^{t_b-t_a} 
\]

\[\equiv 3 \star \quad \Rightarrow \quad (5.31)\]

Mixed position-momentum and momentum-momentum correlations and their graphical representations are given in the Appendix.

The consideration of higher-order correlations with more than one function \( F(\bar{\omega}(t)) \) or \( F(\bar{p}(t)) \) can be reduced to the results (5.9)–(5.12) or (5.15)–(5.18) by expanding them with respect to the classical path or momentum, respectively. By expanding both functions in the expectation value, one obtains, for example,

\[
(F_i(\bar{\omega}(t_1))F_j(\bar{\omega}(t_2)))_{\Omega}^{t_b-t_a} 
\]

\[\equiv \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! n!} f_{1,m} f_{2,n} \langle x^m(t_1)x^n(t_2) \rangle_{\Omega}^{t_b-t_a} \quad (5.33)\]

with

\[
f_{1,m} = F^{(m)}(0), \quad i = 1, 2. \quad (5.34)\]
But constructing graphical rules for such general correlations is more involved due to the various summations over products of powers of propagators \( G_{ij}(t_i,t_j) \) with \( i,j=1,2 \).

Finally, we apply the diagrammatic rules to the anharmonic oscillator with \( \tilde{x}^4 \) interaction, which is a powerful system being discussed in detail with the help of a perturbation expansion ([1], Chap. 3). With the Green functions given by Eqs. (3.22), (3.44), and (3.45), the two-point correlation for the anharmonic system with arbitrary time-dependent frequency can then be expressed graphically, yielding the known decomposition for the second-order perturbative contribution

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle \tilde{x}_i(t_1) \tilde{x}_j(t_2) \rangle_{\tilde{\Omega},\epsilon} = 72 \quad \text{ring} + 24 \quad \text{bubble} \quad (5.35)
\]

with subscript \( \epsilon \) indicating that we restrict to connected graphs only. Beyond this, our theory allows us to describe nonstandard systems with polynomial interactions (5.28) depending on both position and momentum, to higher order. Finally, we want to give the graphs for a four-interaction \( \tilde{x}^2 \tilde{p}^2 \) to the second order to see the variations of possible graphs in comparison with Eq. (5.35):

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle \tilde{x}(t_1) \tilde{p}(t_2) \rangle_{\tilde{\Omega},\epsilon} = 2 \quad \text{ring} + 16 \quad \text{bubble} + 2 \quad \text{dashed bubble} + 4 \quad \text{double bubble} + 4 \quad \text{double dashed bubble} + 16 \quad (5.36)
\]

We see that we have the same class of graphs already occurring in Eq. (5.35), however, with different propagators connecting the vertices. Thus, both classes decay into subclasses with different multiplicities, but the total numbers remain 72 and 24 for each type of class, respectively. Furthermore, all graphs are vacuumlike graphs. Eventually, it is easy to construct the Feynman graphs for polynomial correlations higher than second order by applying Wick’s rule or the Feynman rules given in this section.

**VI. SIMPLIFICATIONS FOR PERIODIC PATHS**

Up to now, we discussed the harmonic time evolution amplitude with arbitrary frequency and external sources \( k(t),i(t) \) and corresponding Green functions fulfilling Dirichlet boundary conditions. In the sense of the quantum-mechanical partition function

\[
Z = \int dx \langle xt_b | xt_a \rangle, \quad (6.1)
\]

which is an integral over the time evolution amplitude for closed paths, it is of interest to investigate the generating functional for closed paths. In analogy to Eq. (6.1), we define

\[
Z[k(t),i(t)] = \int dx \langle x t_b | x t_a \rangle \langle k(t),i(t) \rangle \quad (6.2)
\]

with Eq. (3.42) for \( x_a = x_b = x \). One immediately observes that \( Z = Z[0,0] \). The integral is easily done, giving

\[
\begin{aligned}
Z[k(t),i(t)] &= \frac{1}{\sqrt{D_a(t_b) - D_b(t_a) - 2}} \times \exp \left\{ -\frac{i}{2h} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \right. \\
&\quad \left. \left[ \frac{1}{M} \left( j(t) \tilde{G}_{ij}^{(1)}(t, t') j(t') + j(t) \tilde{G}_{jk}^{(1)}(t, t') k(t') \right) \right. \\
&\quad \left. + k(t) \tilde{G}_{jk}^{(1)}(t, t') j(t') \right. \\
&\quad \left. + M k(t) \tilde{G}_{kk}^{(1)}(t, t') k(t') \right] \right\}. \quad (6.3)
\end{aligned}
\]

The Green functions, expressed with fundamental solutions (3.2) and (3.3), are found to be

\[
\tilde{G}_{ij}^{(1)}(t, t') = \frac{1}{D_{ij}(t_b)} \left[ G_{ij}^{(1)}(t, t') + \frac{1}{a(t_a, t_b)} g(t) g(t') \right], \quad (6.4)
\]

\[
\tilde{G}_{jk}^{(1)}(t, t') = \frac{1}{D_{jk}(t_b)} \left[ G_{jk}^{(1)}(t, t') + \frac{1}{a(t_a, t_b)} g(t) g(t') \right] = G_{jk}^{(1)}(t, t'), \quad (6.5)
\]
\[
G^i_{kk}(t,t') = \frac{1}{D^i_{a}(t_b)} \left[ G^i_{kk}(t,t') + \frac{1}{a(t_a,t_b)} \delta(t) \delta(t') \right],
\]

with
\[
a(t,t') = D^i_{a}(t') - D^i_{b}(t) - 2.
\]

Since the function
\[
g(t) = D^i_{a}(t) + D^i_{b}(t)
\]
is periodic, \(g(t_a) = g(t_b)\), due to conditions (3.2), (3.3), and (3.6), also the Green function \(G^i_{jk}(t,t')\) becomes periodic,
\[
G^i_{jk}(t_a,t') = G^i_{jk}(t_b,t').
\]

In analogy to the harmonic propagator without external sources (4.1), we can define expectation values consisting of \(N\) position-dependent functions and \(M\) momentum-dependent functions by
\[
\langle F_1(x(t_1))F_2(x(t_2))\cdots F_{N+M}(p(t_{N+M})) \rangle_{\Omega}
= \frac{1}{Z} \int \frac{Dx DP}{2\pi\hbar} F_1(x(t_1))F_2(x(t_2))\cdots F_{N+M}(p(t_{N+M}))
\times \exp \left\{ \frac{i}{\hbar} A[p,x;0,0] \right\}.
\]

We remark that the generalization of Wick’s rule and the graphical representation with the help of Feynman diagrams of such correlation functions is exactly the same as given in the preceding section after substituting the Green functions \(G(t,t')\) by \(\tilde{G}(t,t')\) and expectation values (4.1) by Eq. (6.10).

**VIII. SUMMARY AND OUTLOOK**

We have reduced generating functionals with fixed end points to those with vanishing end points by adding special singular sources to the currents. The new generating functionals were calculated explicitly for the harmonic oscillator with time-dependent frequency. From this expression, a smearing formula was derived which serves to calculate correlation functions for arbitrary polynomial or nonpolynomial position- and momentum-dependent couplings. We have further found a generalization of Wick’s theorem of decomposing correlation functions involving functions of the canonical variables of the system. This gives rise to certain generalized Feynman rules for position- and momentum-dependent expectation values.

Due to its universality, the theory should serve as a basis for investigating physical systems with a nonstandard Hamiltonian via perturbation theory and its variational extension. Note that a perturbation theory for momentum-dependent interactions arises in important field theories such as the nonlinear \(\sigma\) model. Our work is supposed to lay the foundation for a more efficient perturbation treatment of such a theory.

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**APPENDIX: GENERALIZED CORRELATION FUNCTIONS**

In this appendix we give the expectations for the correlation between a general position- or momentum-dependent function and a polynomial up to order \(n = 4\).

**Position-momentum coupling**

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1))\bar{p}(t_2) \rangle_{\Omega}^{t_b-t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 iG_{jk}(t_1,t_2)\langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{t_b-t_a}
\equiv \star \quad \star \quad \star
\]

\[
= \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1))\bar{p}^2(t_2) \rangle_{\Omega}^{t_b-t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{kk}(t_1,t_2)\langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{t_b-t_a} + [iG_{jk}(t_1,t_2)]^2\langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{t_b-t_a} \right\}
\equiv \star \quad \star \quad \star + \star \quad \star
\]

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1))\bar{p}^3(t_2) \rangle_{\Omega}^{t_b-t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ 3iG_{kk}(t_1,t_2)iG_{kk}(t_1,t_2)\langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{t_b-t_a} + [iG_{jk}(t_1,t_2)]^3\langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{t_b-t_a} \right\}
\equiv 3 \star \quad \star \quad \star + \star \quad \star
\]
\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\vec{x}(t_1))\vec{p}(t_2) \rangle^{t_b,s}_{t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ [iG_{ik}(t_1,t_2)]^{t_b,s}_{t_a} + 6[iG_{jk}(t_1,t_2)]^{t_b,s}_{t_a} \right. \\
\left. \times \langle F(\vec{x}(t_1)) \rangle^{t_b,s}_{t_a} + [iG_{ik}(t_1,t_2)]^{t_b,s}_{t_a} \right\}^{t_b,s}_{t_a} \equiv \star \bigcirc + 6 \star \bigcirc + \bigcirc \bigcirc \bigcirc.
\]

**Momentum-position coupling**

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\vec{p}(t_1))\vec{x}(t_2) \rangle^{t_b,s}_{t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{ik}(t_1,t_2)\langle F(\vec{p}(t_1)) \rangle^{t_b,s}_{t_a} \right. \\
\left. + [iG_{jk}(t_1,t_2)]^{t_b,s}_{t_a} \right\}^{t_b,s}_{t_a} \equiv \star \bigcirc + \bigcirc \bigcirc \bigcirc.
\]

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\vec{p}(t_1))\vec{x}^2(t_2) \rangle^{t_b,s}_{t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ 3iG_{ik}(t_1,t_2)iG_{jk}(t_1,t_2)\langle F(\vec{p}(t_1)) \rangle^{t_b,s}_{t_a} \\
\right. \\
\left. + [iG_{jk}(t_1,t_2)]^{t_b,s}_{t_a} \right\}^{t_b,s}_{t_a} \equiv 3 \star \bigcirc + \bigcirc \bigcirc \bigcirc.
\]

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\vec{p}(t_1))\vec{x}^3(t_2) \rangle^{t_b,s}_{t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ [iG_{ij}(t_1,t_2)]^{t_b,s}_{t_a} + 6[iG_{jk}(t_1,t_2)]^{t_b,s}_{t_a} \right. \\
\left. \times \langle F(\vec{p}(t_1)) \rangle^{t_b,s}_{t_a} + [iG_{jk}(t_1,t_2)]^{t_b,s}_{t_a} \right\}^{t_b,s}_{t_a} \equiv \star \bigcirc + 6 \star \bigcirc + \bigcirc \bigcirc \bigcirc.
\]

**Momentum-momentum coupling**

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\vec{p}(t_1))\vec{p}(t_2) \rangle^{t_b,s}_{t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{ik}(t_1,t_2)\langle F(\vec{p}(t_1)) \rangle^{t_b,s}_{t_a} \right. \\
\left. + [iG_{ik}(t_1,t_2)]^{t_b,s}_{t_a} \right\}^{t_b,s}_{t_a} \equiv \star \bigcirc,
\]

\[
\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\vec{p}(t_1))\vec{p}^2(t_2) \rangle^{t_b,s}_{t_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{ik}(t_1,t_2)\langle F(\vec{p}(t_1)) \rangle^{t_b,s}_{t_a} + [iG_{ik}(t_1,t_2)]^{t_b,s}_{t_a} \right. \\
\left. \times \langle F(\vec{p}(t_1)) \rangle^{t_b,s}_{t_a} + [iG_{ik}(t_1,t_2)]^{t_b,s}_{t_a} \right\}^{t_b,s}_{t_a} \equiv \star \bigcirc + \bigcirc \bigcirc \bigcirc.
\]
The case of position-position coupling has already been calculated in Sec. V C.


