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Generating functionals for harmonic expectation values of paths with fixed end points: Feynman diagrams for nonpolynomial interactions

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We introduce a general class of generating functionals for the calculation of quantum-mechanical expectation values of arbitrary functionals of fluctuating paths with fixed end points in configuration or momentum space. The generating functionals are calculated explicitly for the harmonic oscillator with time-dependent frequency, and used to derive a smearing formula for correlation functions of polynomial and nonpolynomial functions of time-dependent positions and momenta. This formula summarizes the effect of quantum fluctuations, and serves to derive generalized Wick rules and Feynman diagrams for perturbation expansions of nonpolynomial interactions. [S1063-651X(99)04508-0]

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I. INTRODUCTION

A useful technique for describing compactly the properties of a quantum-mechanical system is to define a suitable generating functional of some external source or current $j(t)$. The desired properties are obtained from functional derivatives with respect to $j(t)$. For example, the correlation functions and the time evolution amplitude in one space dimension x are determined by a generating functional which is a path integral in configuration space over all paths $x(t)$ with fixed end points $x(t_a)=x_a$, $x(t_b)=x_b$ [[1], Chap. 2]:

$$(x_b t_b | x_a t_a)[j(t)] = \int_{x_a,t_a}^{x_b,t_b} \mathcal{D}x(t) \exp\left\{ \frac{i}{\hbar} \mathcal{A}[x(t);j(t)] \right\}, \quad (1.1)$$

where the exponent contains the classical action $\mathcal{A}[x(t)]$ plus a source term linear in $x(t)$:

$$\mathcal{A}[x(t);j(t)] = \mathcal{A}[x(t)] + \int_{t_a}^{t_b} dt\, x(t) j(t). \quad (1.2)$$

In this paper we set up a useful alternative expression for the generating functional (1.1) and a related one in momentum space. This alternative expression is obtained by extending the current $j(t)$ by singular sources proportional to $\dot{\delta}(t_b - t)$ and $\dot{\delta}(t-t_a)$, and by reducing the path integral (1.1) with fixed end points in configuration space to one with vanishing end points. This will permit us to simplify considerably the calculation of quantum-mechanical correlation functions. To see this simplification explicitly, consider a harmonic oscillator whose action reads

$$\mathcal{A}[x(t)] = \int_{t_a}^{t_b} dt \left[\frac{M}{2} \dot{x}^2(t) - \frac{M}{2} \omega^2 x^2(t) \right], \quad (1.3)$$

for which the generating functional can be calculated as follows [[1], Eq. (3.89)]:

$$(x_b t_b | x_a t_a)[j(t)] = \left(\frac{M\omega}{2\pi i \hbar \sin \omega(t_b - t_a)} \right)^{1/2} \exp\left\{ \frac{iM\omega[(x_b^2 + x_a^2)\cos \omega(t_b - t_a) - 2x_a x_b]}{2\hbar \sin \omega(t_b - t_a)} \right\}$$
$$\times \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{x_a \sin \omega(t_b - t) + x_b \sin \omega(t - t_a)}{\sin \omega(t_b - t_a)} j(t) - \frac{i}{\hbar M\omega} \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \frac{\sin \omega(t_b - t) \sin \omega(t' - t_a)}{\sin \omega(t_b - t_a)} j(t) j(t') \right\}. \quad (1.4)$$

The nonzero end points x_a and x_b make this expression quite involved. For vanishing end points, however, it simplifies to

$$(X_b = 0\ \ t_b | x_a = 0\ \ t_a)[j(t)] = \left(\frac{M\omega}{2\pi i \hbar \sin \omega(t_b - t_a)} \right)^{1/2} \times \exp\left\{ -\frac{i}{\hbar M\omega} \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \frac{\sin \omega(t_b - t) \sin \omega(t' - t_a)}{\sin \omega(t_b - t_a)} j(t) j(t') \right\}. \quad (1.5)$$

The observation which motivates the present paper relies on replacing the current $j(t)$ in the simple expression (1.5) by

$$j'(t)=j(t)+Mx_a\dot{\delta}(t-t_a)+Mx_b\dot{\delta}(t_b-t), \qquad (1.6)$$

where the δ functions are understood as $\dot{\delta}(t-t_a+\epsilon)$ and $\dot{\delta}(t_b-\epsilon-t)$ in the limit $\epsilon \to 0$. By performing some partial integrations, this replacement reproduces all terms in the complicated generating functional (1.4), except for a rather trivial additional singular phase factor. The important relation is

$$(x_b t_b|x_a t_a)[j(t)]=(x_b=0\ t_b|x_a=0\ t_a) \times[j(t)+Mx_a\dot{\delta}(t-t_a)+Mx_b\dot{\delta}(t_b-t)] \times\exp\left\{\frac{iM}{2\hbar}(x_b^2+x_a^2)\delta(0)\right\}. \qquad (1.7)$$

In Sec. II we prove that the relation (1.7) holds for an arbitrary quantum-mechanical system whose Hamiltonian has the standard form

$$H_0(p,x,t)=\frac{p^2}{2M}+V(x,t). \qquad (1.8)$$

In Sec. III we calculate explicit amplitudes for a harmonic oscillator with arbitrary time-dependent frequency, and as an important application we derive in Sec. IV a smearing formula for calculating expectation values of polynomial and nonpolynomial functions of time-dependent positions and momenta. In particular, this result would allow us to calculate expectation values appearing in perturbation expansions for nonlinear interactions, as, for example, for the nonlinear σ model. In Sec. V we show that our smearing formula generalizes Wick rules and Feynman diagrams for harmonic expectation values from products of variables to mixtures of nonpolynomial functions and polynomials. In Sec. VI, we finally specialize our generating functional to periodic paths.

II. GENERATING FUNCTIONALS

We begin by setting up phase-space path integrals for generating functionals with fixed end points in either configuration or momentum space. The action contains additional currents $k(t)$ and $j(t)$ coupled linearly to momentum $p(t)$ and position $x(t)$. By extending the currents with singular δ functions as in Eq. (1.6), we reduce the path integrals with fixed end points to those with vanishing end points. Our procedure applies to arbitrary Hamiltonians $H_0(p,x,t)$, with certain simplifications resulting from a standard Hamiltonian (1.8).

A. General phase-space formulation

Consider a quantum-mechanical particle coupled to a momentum and a position source $k(t)$ and $j(t)$ with the classical Hamiltonian

$$H(p,x,t)=H_0(p,x,t)-pk(t)-xj(t), \qquad (2.1)$$

where the corresponding action reads

$$A[p(t),x(t);k(t),j(t)]=\int_{t_a}^{t_b}dt\{p(t)\dot{x}(t)-H(p(t),x(t),t)\}. \qquad (2.2)$$

The total time evolution amplitude between fixed space points x_a and x_b is given by the path integral

$$(x_b t_b|x_a t_a)[k(t),j(t)]=\int_{x_a,t_a}^{x_b,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar} \times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\}. \qquad (2.3)$$

A Fourier transformation with respect to x_a and x_b produces the time evolution amplitude in momentum space,

$$(p_b t_b|p_a t_a)[k(t),j(t)] =\int_{-\infty}^{+\infty}dx_a\int_{-\infty}^{+\infty}dx_b e^{-i(p_b x_b-p_a x_a)/\hbar} \times(x_b t_b|x_a t_b)[k(t),j(t)]. \qquad (2.4)$$

Here the initial and final momenta p_a and p_b are held fixed, so that the right-hand side may be written as the path integral

$$(p_b t_b|p_a t_a)[k(t),j(t)]=\int_{p_a,t_a}^{p_b,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar} \times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\}. \qquad (2.5)$$

We remark that both path integrals (2.3) and (2.5) are properly defined as continuum limits of ordinary integrals after a time-slicing procedure. Since end points of paths are fixed in coordinate and momentum space, respectively, the discretized expressions for the path integrals turn out to be slightly asymmetric in $p(t)$ and $x(t)$ [[1], Chap. 2].

The time evolution amplitudes (2.3) and (2.5) with fixed end points can now be reduced to corresponding ones with vanishing end points. For this, we shift the current $k(t)$ in Eq. (2.1) by a source term $x_b\delta(t_b-t)-x_a\delta(t-t_a)$ and observe that this produces by Eqs. (2.2) and (2.5) an overall phase factor:

$$(p_b t_b|p_a t_a)[k(t)+x_b\delta(t_b-t)-x_a\delta(t-t_a),j(t)] =\exp\left[\frac{i}{\hbar}(p_b x_b-p_a x_a)\right](p_b t_b|p_a t_a)[k(t),j(t)]. \qquad (2.6)$$

By inverting the Fourier transformation (2.4), the configuration space amplitude (2.3) is seen to satisfy

$$(x_bt_b|x_at_a)[k(t)+x_b'\delta(t_b-t)-x_a'\delta(t-t_a),j(t)]$$
$$=(x_b+x_b'\ t_b|x_a+x_a'\ t_a)[k(t),j(t)], \tag{2.7}$$

where again the δ functions are understood as $\delta(t_b-\epsilon-t)$ and $\delta(t-t_a+\epsilon)$ in the limit $\epsilon\to 0$. Because of this relation, the amplitude (2.3) can be reduced to a path integral with vanishing end points but additional δ terms in the current $k(t)$:

$$(x_bt_b|x_at_a)[k(t),j(t)]$$
$$=(x_b=0\ t_b|x_a=0\ t_a)[k(t)+x_b\delta(t_b-t)$$
$$-x_a\delta(t-t_a),j(t)]. \tag{2.8}$$

A similar expression exists, if momentum end points are fixed in momentum space by adding $p_a\delta(t-t_a)-p_b\delta(t_b-t)$ to the current $j(t)$:

$$(p_bt_b|p_at_a)[k(t),j(t)]=(p_b=0\ t_b|p_a=0\ t_a)[k(t),j(t)$$
$$+p_a\delta(t-t_a)-p_b\delta(t_b-t)]. \tag{2.9}$$

We now explore the consequences of these two relations for the calculation of correlation functions.

B. Correlation functionals

The functional dependence of the time evolution amplitudes (2.3) and (2.5) on the currents $k(t)$ and $j(t)$ allows us to calculate expectation values of arbitrary functionals $F[p(t),x(t)]$ from the path integral

$$\langle F[p(t),x(t)]\rangle[k(t),j(t)]_{v_a,t_a}^{v_b,t_b}$$
$$=\frac{1}{(v_bt_b|v_at_a)[k(t),j(t)]}$$
$$\times\int_{v_a,t_a}^{v_b,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar}F[p(t),x(t)]$$
$$\times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\}, \tag{2.10}$$

where the variable v may be p or x. The usual correlation functions

$$\langle p(t_1)\cdots p(t_n)x(t_{n+1})\cdots x(t_m)\rangle[k(t),j(t)]_{v_a,t_a}^{v_b,t_b}$$
$$=\frac{1}{(v_bt_b|v_at_a)[k(t),j(t)]}$$
$$\times\int_{v_a,t_a}^{v_b,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar}p(t_1)\cdots p(t_n)x(t_{n+1})\cdots x(t_m)$$
$$\times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\} \tag{2.11}$$

are special cases of Eq. (2.10), so we shall call the general expectation values (2.10) *correlation functionals*. The sources $k(t)$ and $j(t)$ permit us to express Eq. (2.10) in terms of functional derivatives:

$$\langle F[p(t),x(t)]\rangle[k(t),j(t)]_{v_a,t_a}^{v_b,t_b}$$
$$=\frac{F\left[\frac{\hbar}{i}\frac{\delta}{\delta k(t)},\frac{\hbar}{i}\frac{\delta}{\delta j(t)}\right](v_bt_b|v_at_a)[k(t),j(t)]}{(v_bt_b|v_at_a)[k(t),j(t)]}. \tag{2.12}$$

Recalling Eqs. (2.8) and (2.9), we shall rewrite the functionals $(v_bt_b|v_at_a)[k(t),j(t)]$ in a unified common way as follows:

$$(v_bt_b|v_at_a)[k(t),j(t)]$$
$$=\int_{w_a=0,t_a}^{w_b=0,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar}\,\delta(v(t_a)-v_a)\delta(v(t_b)-v_b)$$
$$\times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\}, \tag{2.13}$$

where the paths $v(t)$ stand either for $p(t)$ or for $x(t)$. In each of these cases, the paths $w(t)$ denote the conjugate variables $x(t)$ or $p(t)$, respectively. In this form, the path integral possesses the advantage that usual correlation functions (2.11) can be determined by path averages, in which intermediate and end points are treated on equal footing. Indeed, inserting δ functions according to

$$\langle p(t_1)\cdots p(t_n)x(t_{n+1})\cdots x(t_m)\rangle[k(t),j(t)]_{v_a,t_a}^{v_b,t_b}$$
$$=\frac{1}{(v_bt_b|v_at_a)[k(t),j(t)]}\int_{-\infty}^{+\infty}dp_1\cdots\int_{-\infty}^{+\infty}dp_n$$
$$\times\int_{-\infty}^{+\infty}dx_{n+1}\cdots\int_{-\infty}^{+\infty}dx_m\ p_1\cdots p_nx_{n+1}\cdots x_m$$
$$\times\int_{v_a,t_a}^{v_b,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar}\,\delta(p(t_1)-p_1)\cdots\delta(p(t_n)-p_n)$$
$$\times\delta(x(t_1)-x_1)\cdots\delta(x(t_m)-x_m)$$
$$\times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\}, \tag{2.14}$$

we obtain with a similar reasoning

$$\langle p(t_1)\cdots p(t_n)x(t_{n+1})\cdots x(t_m)\rangle[k(t),j(t)]_{v_a,t_a}^{v_b,t_b}$$
$$=\frac{1}{(v_bt_b|v_at_a)[k(t),j(t)]}\int_{-\infty}^{+\infty}dp_1\cdots\int_{-\infty}^{+\infty}dp_n$$
$$\times\int_{-\infty}^{+\infty}dx_{n+1}\cdots\int_{-\infty}^{+\infty}dx_m\ p_1\cdots p_nx_{n+1}\cdots x_m$$
$$\times\int_{w_a=0,t_a}^{w_b=0,t_b}\frac{\mathcal{D}p(t)\mathcal{D}x(t)}{2\pi\hbar}\,\delta(v(t_a)-v_a)$$
$$\times\delta(p(t_1)-p_1)\cdots\delta(p(t_n)-p_n)$$
$$\times\delta(x(t_{n+1})-x_{n+1})\cdots\delta(x(t_m)-x_m)\delta(v(t_b)-v_b)$$
$$\times\exp\left\{\frac{i}{\hbar}\mathcal{A}[p(t),x(t);k(t),j(t)]\right\}. \tag{2.15}$$

C. Standard Hamiltonian

The above formalism can be made more specific for the standard Hamiltonian (1.8). Then the path integrals over the momentum paths $p(t)$ in Eqs. (2.3) and (2.5) become harmonic and can be explicitly evaluated. The phase-space integral (2.3), for instance, reduces to the configuration space path integral,

$$(x_b t_b|x_a t_a)[k(t),j(t)]=\int_{x_a,t_a}^{x_b,t_b}\mathcal{D}x(t)\times\exp\left\{\frac{i}{\hbar}\mathcal{A}[x(t);k(t),j(t)]\right\}, \tag{2.16}$$

where the current $k(t)$ couples linearly to the path momentum $M\dot{x}(t)$ in the action

$$\mathcal{A}[x(t);k(t),j(t)]=\int_{t_a}^{t_b}dt\left\{\frac{M}{2}\dot{x}^2(t)-V(x(t),t)+x(t)j(t)+M\dot{x}(t)k(t)+\frac{M}{2}k^2(t)\right\}. \tag{2.17}$$

A subsequent partial integration transforms the current $k(t)$ to an effective coordinate current with an extra phase factor:

$$(x_b t_b|x_a t_a)[k(t),j(t)]=(x_b t_b|x_a t_a)[0,j(t)-M\dot{k}(t)]\times\exp\left\{\frac{iM}{\hbar}\left[x_b k_b-x_a k_a+\frac{1}{2}\int_{t_a}^{t_b}dt\,k^2(t)\right]\right\}. \tag{2.18}$$

Note that combining Eqs. (2.8) and (2.18) proves relation (1.7) for any arbitrary quantum-mechanical system with the standard Hamiltonian (1.8).

In the next section we determine the generating functional $(x_b t_b|x_a t_a)[0,j(t)]$ for a harmonic oscillator with arbitrary time-dependent frequency $\Omega(t)$ and use Eq. (2.18) to construct the full generating functional $(x_b t_b|x_a t_a)[k(t),j(t)]$.

III. TIME-DEPENDENT HARMONIC OSCILLATOR

Consider a standard Hamiltonian (1.8) with a harmonic potential containing an arbitrary time-dependent frequency:

$$V(x,t)=\frac{M}{2}\Omega^2(t)x^2. \tag{3.1}$$

The generating functionals (2.3) and (2.5) are then expressable in terms of two fundamental solutions $D_a(t),D_b(t)$ of the corresponding classical equation of motion with particular boundary conditions [2]

$$\hat{K}(t)D_a(t)=0;\quad D_a(t_a)=0,\ \dot{D}_a(t_a)=1, \tag{3.2}$$

$$\hat{K}(t)D_b(t)=0;\quad D_b(t_b)=0,\ \dot{D}_b(t_b)=-1, \tag{3.3}$$

where $\hat{K}(t)$ denotes the operator

$$\hat{K}(t)=-\partial_t^2-\Omega^2(t). \tag{3.4}$$

Since the time derivative of the Wronski determinant

$$W(t)=D_a(t)\dot{D}_b(t)-\dot{D}_a(t)D_b(t) \tag{3.5}$$

vanishes, we observe the identity

$$D_a(t_b)=D_b(t_a). \tag{3.6}$$

Note that a similar identity does not hold for the time derivatives of the two fundamental solutions $D_a(t)$ and $D_b(t)$. Indeed, partially integrating the differential equation for $\dot{D}_a(t)$ and taking into account Eqs. (3.2)–(3.5), we deduce

$$\dot{D}_b(t_a)+\dot{D}_a(t_b)=-2\int_{t_a}^{t_b}dt\,\Omega(t)\dot{\Omega}(t)D_a(t)D_b(t). \tag{3.7}$$

Let us now determine the time evolution amplitude (2.16) in configuration space for a vanishing current $k(t)$. We decompose the paths $x(t)$ into the classical path $x_{\rm cl}^j(t)$ and the quantum fluctuations $\delta x(t)$ around it:

$$x(t)=x_{\rm cl}^j(t)+\delta x(t). \tag{3.8}$$

The classical path $x_{\rm cl}^j(t)$ solves the boundary value problem

$$\hat{K}(t)x_{\rm cl}^j(t)=-\frac{j(t)}{M};\quad x_{\rm cl}^j(t_a)=x_a,\ x_{\rm cl}^j(t_b)=x_b, \tag{3.9}$$

and the fluctuations $\delta x(t)$ vanish at the end points:

$$\delta x(t_a)=\delta x(t_b)=0. \tag{3.10}$$

Inserting the decomposition (3.8) into the action (2.17), we observe that due to Eqs. (3.9) and (3.10) the total action decomposes into a classical part

$$\mathcal{A}[x_{\rm cl}^j(t);0,j(t)]=\frac{M}{2}[x_b\dot{x}_{\rm cl}^j(t_b)-x_a\dot{x}_{\rm cl}^j(t_a)]+\frac{1}{2}\int_{t_a}^{t_b}dt\,x_{\rm cl}^j(t)j(t), \tag{3.11}$$

and a fluctuation part, which is simply the classical action evaluated for the fluctuations $\delta x(t)$ at $j=0$:

$$\mathcal{A}[x(t);0,j(t)]=\mathcal{A}[x_{\rm cl}^j(t);0,j(t)]+\mathcal{A}[\delta x(t);0,0]. \tag{3.12}$$

Inserting this into the original path integral (2.16), it factorizes into the product of a classical amplitude with the classical action (3.11), and an additional fluctuation factor which is equal to the amplitude at vanishing end points:

$$(x_b t_b|x_a t_a)[0;j(t)]=\exp\left\{\frac{i}{\hbar}\mathcal{A}[x_{\rm cl}^j(t);0,j(t)]\right\}\times(x_b=0\ t_b|x_a=0\ t_a)[0,0]. \tag{3.13}$$

A. Classical action

The classical action in the presence of currents can be expressed in terms of the solutions $D_a(t),D_b(t)$ of the time-dependent harmonic boundary value problems (3.2) and (3.3). First we decompose the solution of the boundary value problem (3.9) in the presence of external sources into a homogeneous and an inhomogeneous contribution:

$$x_{\rm cl}^j(t)=x_{\rm cl}(t)+\Delta x_{\rm cl}^j(t). \tag{3.14}$$

The homogeneous solution reads

$$x_{\rm cl}(t)=\frac{D_b(t)x_a+D_a(t)x_b}{D_a(t_b)}, \tag{3.15}$$

while the inhomogeneous one is given by

$$\Delta x_{\rm cl}^j(t)=-\frac{1}{M}\int_{t_a}^{t_b}dt'\,G_{jj}^x(t,t')j(t'), \tag{3.16}$$

where $G_{jj}^x(t,t')$ denotes the Green function of the classical equation of motion

$$\hat{K}(t)G_{jj}^x(t,t')=\delta(t-t') \tag{3.17}$$

with Dirichlet boundary conditions

$$G_{jj}^x(t_a,t')=G_{jj}^x(t_b,t')=0. \tag{3.18}$$

From Eq. (3.17) we deduce that the Green function $G_{jj}^x(t,t')$ solves the homogeneous differential equation for $t\neq t'$:

$$\hat{K}(t)G_{jj}^x(t,t')=0, \tag{3.19}$$

and that its first derivative $\partial_t G_{jj}^x(t,t')$ is discontinuous at $t=t'$:

$$\lim_{\epsilon\downarrow 0}\,[\partial_t G_{jj}^x(t,t')|_{t=t'+\epsilon}-\partial_t G_{jj}^x(t,t')|_{t=t'-\epsilon}]=-1. \tag{3.20}$$

The Green function itself is continuous around $t=t'$:

$$\lim_{\epsilon\downarrow 0}\,[G_{jj}^x(t,t')|_{t=t'+\epsilon}-G_{jj}^x(t,t')|_{t=t'-\epsilon}]=0. \tag{3.21}$$

The solution of Eqs. (3.18)–(3.21) is given by Wronski's famous expression

$$G_{jj}^x(t,t')=\frac{\Theta(t-t')D_b(t)D_a(t')+\Theta(t'-t)D_a(t)D_b(t')}{D_a(t_b)}=G_{jj}^x(t',t), \tag{3.22}$$

where $\Theta(t-t')$ denotes the Heaviside function which vanishes for $t<t'$ and is equal to unity for $t>t'$. Inserting Eqs. (3.14) and (3.16) we obtain for the classical action (3.11)

$$\mathcal{A}[x_{\rm cl}^j(t);0,j(t)]=\frac{M}{2D_a(t_b)}[\dot{D}_a(t_b)x_b^2-\dot{D}_b(t_a)x_a^2-2x_ax_b]+\int_{t_a}^{t_b}dt\,x_{\rm cl}(t)j(t)-\frac{1}{2M}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\,G_{jj}^x(t,t')j(t)j(t'), \tag{3.23}$$

where $x_{\rm cl}(t)$ and $G_{jj}^x(t,t')$ are given by Eqs. (3.15) and (3.22), respectively.

B. Fluctuation factor

Now we calculate the fluctuation factor in Eq. (3.13). Recalling the path representation (2.16) with the action (2.17), we have to evaluate

$$(x_b=0t_b|x_a=0t_a)[0,0]=\int_{\delta x_a=0,t_a}^{\delta x_b=0,t_b}\mathcal{D}\delta x(t)\exp\left[\frac{iM}{2\hbar}\int_{t_a}^{t_b}dt\;\delta x(t)\hat{K}(t)\delta x(t)\right]. \tag{3.24}$$

To this end we decompose the fluctuation $\delta x(t)$ in Eq. (3.24) into eigenfunctions $x_n(t)$ of the operator $\hat{K}(t)$ in Eq. (3.4) with Dirichlet boundary conditions

$$\hat{K}(t)x_n(t)=\lambda_n x_n(t);\quad x_n(t_a)=x_n(t_b)=0 \tag{3.25}$$

which satisfy the orthonormality and completeness relations

$$\int_{t_a}^{t_b}dt\,x_n(t)x_{n'}(t)=\delta_{n,n'}, \tag{3.26}$$

$$\sum_n x_n(t)x_n(t')=\delta(t-t'), \tag{3.27}$$

as follows:

$$\delta x(t)=\sum_n c_nx_n(t). \tag{3.28}$$

The path integral over all possible fluctuations $\delta x(t)$ in Eq. (3.24) amounts to a product of integrals over all expansion coefficients c_n:

$$\int_{\delta x_a=0,t_a}^{\delta x_b=0,t_b}\mathcal{D}\delta x(t)=J\left\{\prod_n\int_{-\infty}^{+\infty}dc_n\right\}. \tag{3.29}$$

The Jacobi determinant J of the transformation (3.28) is an irrelevant constant. Applying Eqs. (3.25)–(3.29), the path integral (3.24) is finally determined by

$$(x_b=0\;t_b|x_a=0\;t_a)[0,0]=\frac{J}{\sqrt{\det\hat{K}(t)}}, \tag{3.30}$$

where the determinant of the operator $\hat{K}(t)$ is equal to the product of its eigenvalues

$$\det \hat{K}(t) = \prod_n \lambda_n . \tag{3.31}$$

C. Operator determinant

In order to calculate the operator determinant (3.31), it is advantageous to introduce a one-parameter family of operators [3,4]

$$\hat{K}^g(t) = -\partial_t^2 - g\Omega^2(t), \tag{3.32}$$

depending linearly on a coupling strength parameter $g \in [0,1]$, and coinciding with the original operator $\hat{K}(t)$ in Eq. (3.4) for $g=1$. It is possible to relate the operator determinant $\det \hat{K}^g(t)$ to the fundamental solutions $D_a^g(t)$, $D_b^g(t)$, and to the Green function $G_{jj}^{x;g}(t,t')$ emerging from Eqs. (3.2), (3.3), (3.17), and (3.18). For this we substitute the operator $\hat{K}(t)$ by $\hat{K}^g(t)$, and differentiate the g-dependent version of the eigenvalue problem (3.25) with respect to g:

$$\hat{K}^g(t)\frac{\partial x_n^g(t)}{\partial g} - \Omega^2(t) x_n^g(t) = \frac{\partial \lambda_n^g}{\partial g} x_n^g(t) + \lambda_n^g \frac{\partial x_n^g(t)}{\partial g}. \tag{3.33}$$

Multiplying Eq. (3.33) with $x_n^g(t)/\lambda_n^g$ and performing a summation over n plus an integration with respect to t, we obtain with Eqs. (3.25), (3.26), and (3.31),

$$\frac{\partial}{\partial g}\ln \det \hat{K}^g(t) = -\int_{t_a}^{t_b} dt\, \Omega^2(t) G_{jj}^{x;g}(t,t). \tag{3.34}$$

In the last step we have used the spectral decomposition of the Green function

$$G_{jj}^{x,g}(t,t') = \sum_n \frac{x_n^g(t) x_n^g(t')}{\lambda_n^g}. \tag{3.35}$$

To solve the differential equation (3.34), we differentiate the boundary value equation (3.2) for $D_a^g(t)$ with respect to g, and obtain the inhomogeneous initial value problem

$$\hat{K}^g(t)\frac{\partial D_a^g(t)}{\partial g} = \Omega^2(t) D_a^g(t);$$

$$\left.\frac{\partial D_a^g(t)}{\partial g}\right|_{t=t_a} = \frac{\partial}{\partial t}\left.\frac{\partial D_a^g(t)}{\partial g}\right|_{t=t_a} = 0. \tag{3.36}$$

Generalizing Eq. (3.22) from $g=1$ to arbitrary values $g \in [0,1]$, the solution of Eq. (3.36) is given by

$$\frac{\partial}{\partial g}\ln D_a^g(t_b) = -\int_{t_a}^{t_b} dt\, \Omega^2(t) G_{jj}^{x;g}(t,t). \tag{3.37}$$

This shows that Eq. (3.34) is solved by

$$\det \hat{K}^g(t) = C D_a^g(t_b), \tag{3.38}$$

where C denotes some constant. Due to this result, the ratio of two fluctuation factors (3.30) with two different parameters g_1 and g_2 can be rewritten as

$$\frac{(x_b=0\ t_b|x_a=0\ t_a)[0,0]^{g_1}}{(x_b=0\ t_b|x_a=0\ t_a)[0,0]^{g_2}} = \left(\frac{D_a^{g_2}(t_b)}{D_a^{g_1}(t_b)}\right)^{1/2}. \tag{3.39}$$

This serves to determine the fluctuation factor of the initial time-dependent harmonic oscillator at $g_1=1$ in terms of the fluctuation factor of the free particle $g_2=0$. The latter is well known and may be calculated explicitly, for instance, via time slicing [[1], Chap. 2] as

$$(x_b=0\ t_b|x_a=0\ t_a)[0,0]^{g_2=0} = \left(\frac{M}{2\pi i\hbar(t_b-t_a)}\right)^{1/2}. \tag{3.40}$$

Since the obvious solution of Eq. (3.2) at $g_2=0$ reads $D_a^{g_2=0}(t_b) = t_b - t_a$, we obtain the famous Gelfand-Yaglom formula for Dirichlet boundary conditions [5]:

$$(x_b=0\ t_b|x_a=0\ t_a)[0,0] = \left(\frac{M}{2\pi i\hbar D_a(t_b)}\right)^{1/2}. \tag{3.41}$$

Note that similar results can also be derived for periodic and antiperiodic boundary conditions [3,4].

D. Full generating functional

Having obtained the generating functional $(x_b t_b|x_a t_a)[0,j(t)]$ of the harmonic oscillator with arbitrary frequency with vanishing current $k(t)$, we now make use of the relation (2.18) to derive the full generating functional $(x_b t_b|x_a t_a)[k(t),j(t)]$. The terms containing the current velocity $\dot{k}(t)$ can be turned into functionals of $k(t)$ itself with the help of several partial integrations. These turn out to remove the extra phase factor in Eq. (2.18). As a result, the time evolution amplitude in the configuration representation is determined by a Van Vleck–Pauli-Morette type of formula [[1], Chap. 4],

$$\begin{aligned}(x_b t_b|x_a t_a)[k(t),j(t)] &= \left(\frac{i}{2\pi\hbar}\frac{\partial^2 \mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)]}{\partial x_b \partial x_a}\right)^{1/2}\\ &\quad\times\exp\left\{\frac{i}{\hbar}\mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)]\right\}\end{aligned} \tag{3.42}$$

with the action

$$\begin{aligned}\mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)] &= \frac{M[\dot{D}_a(t_b)x_b^2 - \dot{D}_b(t_a)x_a^2 - 2x_a x_b]}{2D_a(t_b)} + \int_{t_a}^{t_b} dt\, [x_{\rm cl}(t)j(t)\\ &\quad + p_{\rm cl}(t)k(t)] - \frac{1}{2}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'\Bigg[\frac{1}{M}G_{jj}^x(t,t')j(t)j(t')\\ &\quad + G_{jk}^x(t,t')j(t)k(t') + G_{kj}^x(t,t')k(t)j(t')\\ &\quad + MG_{kk}^x(t,t')k(t)k(t')\Bigg].\end{aligned} \tag{3.43}$$

The homogeneous classical solution $x_{\rm cl}(t)$ is given in Eq.

(3.15), and $p_{\rm cl}(t)$ denotes the classical momentum $p_{\rm cl}(t) \equiv M\dot{x}_{\rm cl}(t)$. The Green function $G^x_{jj}(t,t')$ is given by Eq. (3.22), while the others are

$$G^x_{jk}(t,t') = \frac{\Theta(t-t')D_b(t)\dot{D}_a(t') + \Theta(t'-t)D_a(t)\dot{D}_b(t')}{D_a(t_b)} = G^x_{kj}(t',t), \tag{3.44}$$

$$G^x_{kk}(t,t') = \frac{\Theta(t-t')\dot{D}_b(t)\dot{D}_a(t') + \Theta(t'-t)\dot{D}_a(t)\dot{D}_b(t')}{D_a(t_b)} = G^x_{kk}(t',t). \tag{3.45}$$

By differentiating Eq. (3.43) functionally with respect to j and k, we see that the Green functions correspond to the correlation functions

$$\langle \tilde{x}(t)\tilde{x}(t')\rangle[0,0]^{x_b,t_b}_{x_a,t_a} = \frac{i\hbar}{M} G^x_{jj}(t,t'), \tag{3.46}$$

$$\langle \tilde{x}(t)\tilde{p}(t')\rangle = [0,0]^{x_b,t_b}_{x_a,t_a} = i\hbar G^x_{jk}(t,t') = i\hbar G^x_{kj}(t',t), \tag{3.47}$$

$$(\tilde{p}(t)\tilde{p}(t')\rangle[0,0]^{x_b,t_b}_{x_a,t_a} = i\hbar M G^x_{kk}(t,t') \tag{3.48}$$

with $\tilde{x}(t) = x(t) - x_{\rm cl}(t)$ and $\tilde{p}(t) = p(t) - p_{\rm cl}(t)$. These results can be summarized by the mnemonic rule that the Green functions involving a momentum current $k(t)$ once or twice follow from $G^x_{jj}(t,t')$ by one or two time derivatives if the time derivatives of the Heaviside functions are neglected:

$$G^x_{jk}(t,t') = \frac{''\partial G^x_{jj}(t,t')''}{\partial t'}, \quad G^x_{kj}(t,t') = \frac{''\partial G^x_{jj}(t,t')''}{\partial t},$$

$$G^x_{kk}(t,t') = \frac{''\partial^2 G^x_{jj}(t,t')''}{\partial t \partial t'}. \tag{3.49}$$

A complete analogous expression to Eq. (3.43) is found for the time evolution amplitude in the momentum representation. The Fourier transformation (2.4) of Eq. (3.42) yields a Van Vleck–Pauli Morette type of formula

$$(p_b t_b | p_a t_a)[k(t),j(t)] = \left(2\pi i\hbar \frac{\partial^2 \mathcal{A}(p_b,t_b;p_a,t_a)[k(t),j(t)]}{\partial p_a \partial p_a}\right)^{1/2} \times \exp\left\{\frac{i}{\hbar}\mathcal{A}(p_b,t_b;p_a,t_a)[k(t),j(t)]\right\}, \tag{3.50}$$

where the action is the Legendre transform of Eq. (3.43),

$$\mathcal{A}(p_b,t_b;p_a,t_a)[k(t),j(t)] = \mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)] - p_b x_b + p_a x_a, \tag{3.51}$$

calculated for the conjugate variables

$$p_b = \frac{\partial \mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)]}{\partial x_b},$$

$$p_a = -\frac{\partial \mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)]}{\partial x_a}. \tag{3.52}$$

This brings Eq. (3.51) to the form

$$\begin{aligned}\mathcal{A}(p_b,t_b;p_a,t_a)[k(t),j(t)] &= \frac{D_a(t_b)[\dot{D}_a(t_b)p_a^2 - \dot{D}_b(t_a)p_b^2 - 2p_a p_b]}{2M[1+\dot{D}_a(t_b)\dot{D}_b(t_a)]} + \int_{t_a}^{t_b} dt\, [\overline{x}_{\rm cl}(t)j(t) + \overline{p}_{\rm cl}(t)k(t)] \\ &- \frac{1}{2}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \bigg[\frac{1}{M}G^p_{jj}(t,t')j(t)j(t') + G^p_{jk}(t,t')j(t)k(t') + G^p_{kj}(t,t')k(t)j(t') \\ &+ MG^p_{kk}(t,t')k(t)k(t')\bigg],\end{aligned} \tag{3.53}$$

where the classical solution now reads

$$\overline{x}_{\rm cl}(t) = \frac{p_a[D_a(t) + D_b(t)\dot{D}_a(t_b)] + p_b[D_a(t)\dot{D}_b(t_a) - D_b(t)]}{M[1+\dot{D}_a(t_b)\dot{D}_b(t_a)]}, \tag{3.54}$$

and $\overline{p}_{\rm cl}(t)$ denotes the associated classical momentum $\overline{p}_{\rm cl}(t) \equiv M\dot{\overline{x}}_{\rm cl}(t)$. The Green functions in Eq. (3.53) turn out to be

$$\begin{aligned}G^p_{jj}(t,t') &= \Theta(t-t')\frac{[D_b(t)\dot{D}_a(t_b) + D_a(t)][D_a(t')\dot{D}_b(t_a) - D_b(t')]}{D_a(t_b)[1+\dot{D}_p(t_b)\dot{D}(t_a)]} \\ &+ \Theta(t'-t)\frac{[D_a(t)\dot{D}_b(t_a) - D_b(t)][D_b(t')\dot{D}_a(t_b) + D_a(t')]}{D_a(t_b)[1+\dot{D}_a(t_b)\dot{D}_b(t_a)]} = G^p_{jj}(t',t),\end{aligned} \tag{3.55}$$

$$G^p_{jk}(t,t')=\Theta(t-t')\,\frac{[D_b(t)\dot D_a(t_b)+D_a(t)][\dot D_a(t')\dot D_b(t_a)-\dot D_b(t')]}{D_a(t_b)[1+\dot D_a(t_b)\dot D_b(t_a)]}$$
$$+\Theta(t'-t)\,\frac{[D_a(t)\dot D_b(t_a)-D_b(t)][\dot D_b(t')\dot D_a(t_b)+\dot D_a(t')]}{D_a(t_b)[1+\dot D_a(t_b)\dot D_b(t_a)]}=G^p_{kj}(t',t), \tag{3.56}$$

$$G^p_{kk}(t,t')=\Theta(t-t')\,\frac{[\dot D_b(t)\dot D_a(t_b)+\dot D_a(t)][\dot D_a(t')\dot D_b(t_a)-\dot D_b(t')]}{D_a(t_b)[1+\dot D_a(t_b)\dot D_b(t_a)]}$$
$$+\Theta(t'-t)\,\frac{[\dot D_a(t)\dot D_b(t_a)-\dot D_b(t)][\dot D_b(t')\dot D_a(t_b)+\dot D_a(t')]}{D_a(t_b)[1+\dot D_a(t_b)\dot D_b(t_a)]}=G^p_{kk}(t',t). \tag{3.57}$$

The relation to the correlation functions is similar to Eqs. (3.46)–(3.48):

$$\langle\tilde x(t)\tilde x(t')\rangle[0,0]^{p_b,t_b}_{p_a,t_a}=\frac{i\hbar}{M}G^p_{jj}(t,t'), \tag{3.58}$$

$$\langle\tilde x(t)\tilde p(t')\rangle[0,0]^{p_b,t_b}_{p_a,t_a}=i\hbar G^p_{jk}(t,t')=i\hbar G^p_{kj}(t',t), \tag{3.59}$$

$$\langle\tilde p(t)\tilde p(t')\rangle[0,0]^{p_b,t_b}_{p_a,t_a}=i\hbar M G^p_{kk}(t,t'), \tag{3.60}$$

with $\tilde x(t)=x(t)-\overline{x_e}(t)$ and $\tilde p(t)=p(t)-\overline{p_e}(t)$. The relation between the similar-looking actions (3.43) and (3.53) becomes more transparent by reexpressing both in terms of partial derivatives of the classical solutions $x_{\rm cl}(t),\overline{x_{\rm cl}}(t),p_{\rm cl}(t),\overline{p_{\rm cl}}(t)$ with respect to the end points x_b,x_a and p_b,p_a, respectively. In the configuration representation we obtain

$$\mathcal{A}(x_b,t_b;x_a,t_a)[k(t),j(t)]=\frac{1}{2}\,(x_b,x_a)\begin{pmatrix}\frac{\partial p_b}{\partial x_b} & \frac{\partial p_b}{\partial x_a}\\ -\frac{\partial p_a}{\partial x_b} & -\frac{\partial p_a}{\partial x_a}\end{pmatrix}\begin{pmatrix}x_b\\ x_a\end{pmatrix}+\int_{t_a}^{t_b}dt(x_b,x_a)\begin{pmatrix}\frac{\partial p_{\rm cl}(t)}{\partial x_b} & \frac{\partial x_{\rm cl}(t)}{\partial x_b}\\ \frac{\partial p_{\rm cl}(t)}{\partial x_a} & \frac{\partial x_{\rm cl}(t)}{\partial x_a}\end{pmatrix}\begin{pmatrix}k(t)\\ j(t)\end{pmatrix}$$
$$-\frac{1}{2}\,\frac{\partial x_b}{\partial p_a}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'(k(t),j(t))\left[\Theta(t-t')\begin{pmatrix}\frac{\partial p_{\rm cl}(t)}{\partial x_a}\frac{\partial p_{\rm cl}(t')}{\partial x_b} & \frac{\partial p_{\rm cl}(t)}{\partial x_a}\frac{\partial x_{\rm cl}(t')}{\partial x_b}\\ \frac{\partial x_{\rm cl}(t)}{\partial x_a}\frac{\partial p_{\rm cl}(t')}{\partial x_b} & \frac{\partial x_{\rm cl}(t)}{\partial x_a}\frac{\partial x_{\rm cl}(t')}{\partial x_b}\end{pmatrix}\right.$$
$$\left.+\Theta(t'-t)\begin{pmatrix}\frac{\partial p_{\rm cl}(t)}{\partial x_b}\frac{\partial p_{\rm cl}(t')}{\partial x_a} & \frac{\partial p_{\rm cl}(t)}{\partial x_b}\frac{\partial x_{\rm cl}(t')}{\partial x_a}\\ \frac{\partial x_{\rm cl}(t)}{\partial x_b}\frac{\partial p_{\rm cl}(t')}{\partial x_a} & \frac{\partial x_{\rm cl}(t)}{\partial x_b}\frac{\partial x_{\rm cl}(t')}{\partial x_a}\end{pmatrix}\right]\begin{pmatrix}k(t')\\ j(t')\end{pmatrix}. \tag{3.61}$$

The momentum representation, on the other hand, has the analogous form with x and p interchanged:

$$\mathcal{A}(p_b,t_b;p_a,t_a)[k(t),j(t)]=\frac{1}{2}\,(p_b,p_a)\begin{pmatrix}-\frac{\partial x_b}{\partial p_b} & -\frac{\partial x_b}{\partial p_a}\\ \frac{\partial x_a}{\partial p_b} & \frac{\partial x_a}{\partial p_a}\end{pmatrix}\begin{pmatrix}p_b\\ p_a\end{pmatrix}+\int_{t_a}^{t_b}dt\;(p_b,p_a)\begin{pmatrix}\frac{\partial\overline{p_{\rm cl}}(t)}{\partial p_b} & \frac{\partial\overline{x_{\rm cl}}(t)}{\partial p_b}\\ \frac{\partial\overline{p_{\rm cl}}(t)}{\partial p_a} & \frac{\partial\overline{x_{\rm cl}}(t)}{\partial p_a}\end{pmatrix}\begin{pmatrix}k(t)\\ j(t)\end{pmatrix}$$
$$-\frac{1}{2}\,\frac{\partial x_b}{\partial p_a}\int_{t_a}^{t_b}dt\int_{t_a}^{t_b}dt'(k(t),j(t))\left[\Theta(t-t')\begin{pmatrix}\frac{\partial\overline{p_{\rm cl}}(t)}{\partial p_a}\frac{\partial\overline{p_{\rm cl}}(t')}{\partial p_b} & \frac{\partial\overline{p_{\rm cl}}(t)}{\partial p_a}\frac{\partial\overline{x_{\rm cl}}(t')}{\partial p_b}\\ \frac{\partial\overline{x_{\rm cl}}(t)}{\partial p_a}\frac{\partial\overline{p_{\rm cl}}(t')}{\partial p_b} & \frac{\partial\overline{x_{\rm cl}}(t)}{\partial p_a}\frac{\partial\overline{x_{\rm cl}}(t')}{\partial p_b}\end{pmatrix}\right.$$
$$\left.+\Theta(t'-t)\begin{pmatrix}\frac{\partial\overline{p_{\rm cl}}(t)}{\partial p_b}\frac{\partial\overline{p_{\rm cl}}(t')}{\partial p_a} & \frac{\partial\overline{p_{\rm cl}}(t)}{\partial p_b}\frac{\partial\overline{x_{\rm cl}}(t')}{\partial p_a}\\ \frac{\partial\overline{x_{\rm cl}}(t)}{\partial p_b}\frac{\partial\overline{p_{\rm cl}}(t')}{\partial p_a} & \frac{\partial\overline{x_{\rm cl}}(t)}{\partial p_b}\frac{\partial\overline{x_{\rm cl}}(t')}{\partial p_a}\end{pmatrix}\right]\begin{pmatrix}k(t')\\ j(t')\end{pmatrix}. \tag{3.62}$$

These expressions for the generating functionals (3.42) and (3.50) exhibit clearly the symmetry properties (2.8) and (2.9).

IV. SMEARING FORMULA FOR HARMONIC FLUCTUATIONS

As a first application of the generating functional (3.42) we derive a general rule for calculating correlation functions of polynomial or nonpolynomial functions of $x(t)$ and $p(t)$. The result will be expressed in the form of a *smearing formula*. This formula will represent an essential tool for calculating perturbation expansions with nonpolynomial interactions. Such expansions serve in variational perturbation theory to obtain convergent approximations for quantum-statistical partition functions [6] or density matrices [7].

Consider the correlation functions of a product of local functions for vanishing currents,

$$\langle F_1(x(t_1))F_2(x(t_2))\cdots F_N(x(t_N))F_{N+1}(p(t_{N+1}))F_{N+2}(p(t_{N+2}))\cdots F_{N+M}(p(t_{N+M}))\rangle_\Omega^{x_b,x_a}$$
$$=\frac{1}{(x_bt_b|x_at_a)}\int_{x_a,t_a}^{x_b,t_b}\frac{\mathcal{D}x\mathcal{D}p}{2\pi\hbar}\prod_{n=1}^{N}[F_n(x(t_n))]\prod_{m=1}^{M}[F_{N+m}(p(t_{N+m}))]\exp\left\{\frac{i}{\hbar}\mathcal{A}[p,x;0,0]\right\}, \tag{4.1}$$

where the harmonic time evolution amplitude with zero external currents $(x_bt_b|x_at_a)[0,0]$ is written as $(x_bt_b|x_at_a)$. By Fourier transforming the functions $F_n(x(t_n))$ and $F_{N+m}(p(t_{N+m}))$ according to

$$F_n(x(t_n))=\int_{-\infty}^{+\infty}dx_nF_n(x_n)\,\delta(x_n-x(t_n))=\int_{-\infty}^{+\infty}dx_nF(x_n)\int_{-\infty}^{+\infty}\frac{d\xi_n}{2\pi}\exp\{i\xi_n[x_n-x(\tau_n)]\} \tag{4.2}$$

and

$$F_{N+m}(p(t_{N+m}))=\int_{-\infty}^{+\infty}\frac{dp_m}{2\pi\hbar}F_{N+m}(p_m)\,\delta(p_m-p(t_{N+m}))=\int_{-\infty}^{+\infty}\frac{dp_m}{2\pi\hbar}F_{N+m}(p_m)\int_{-\infty}^{+\infty}d\kappa_me^{-i\kappa_m[p_m-p(t_{N+m})]/\hbar}, \tag{4.3}$$

the correlation functions (4.1) may be reexpressed as

$$\langle F_1(x(t_1))\cdots F_{N+M}(p(t_{N+M}))\rangle_\Omega^{x_b,x_a}=\frac{1}{(x_bt_b|x_at_a)}\prod_{n=1}^{N}\left[\int_{-\infty}^{+\infty}dx_nF_n(x_n)\int_{-\infty}^{+\infty}\frac{d\xi_n}{2\pi}e^{i\xi_nx_n}\right]$$
$$\times\prod_{m=1}^{M}\left[\int_{-\infty}^{+\infty}\frac{dp_m}{2\pi\hbar}F_{N+m}(p_m)\int_{-\infty}^{+\infty}d\kappa_me^{-i\kappa_mp_m/\hbar}\right](x_bt_b|x_at_a)[k,j], \tag{4.4}$$

where the generating functional is given by Eq. (3.42). The currents $j(t)$ and $k(t)$ are specialized to

$$j(t)=-\hbar\sum_{n=1}^{N}\xi_n\delta(t-t_n),\qquad k(t)=\sum_{m=1}^{M}\kappa_m\delta(t-t_{N+m}). \tag{4.5}$$

Inserting these equations into the action (3.43) and the Green functions (3.22), (3.44), and (3.45), we find the Fourier decomposition of the generating functional (3.42), so that the correlation functions (4.4) become

$$\langle F_1(x(t_1))\cdots F_{N+M}(p(t_{N+M}))\rangle_\Omega^{x_b,x_a}$$
$$=\prod_{n=1}^{N}\left[\int_{-\infty}^{+\infty}dx_nF_n(x_n)\int_{-\infty}^{+\infty}\frac{d\xi_n}{2\pi}e^{i\xi_n[x_n-x_{\rm cl}(t_n)]}\right]\prod_{m=1}^{M}\left[\int_{-\infty}^{+\infty}\frac{dp_m}{2\pi\hbar}F_{N+m}(p_m)\int_{-\infty}^{+\infty}d\kappa_me^{-i\kappa_m[p_m-p_{\rm cl}(t_{N+m})]/\hbar}\right]$$
$$\times\exp\left\{-\frac{i\hbar}{2M}\sum_{n,n'=1}^{N}\xi_nG_{jj}^{n,n'}\xi_{n'}+i\sum_{n=1}^{N}\sum_{m=1}^{M}\xi_nG_{jk}^{nm}\kappa_m-\frac{iM}{2\hbar}\sum_{m,m'=1}^{M}\kappa_mG_{kk}^{mm'}\kappa_{m'}\right\}, \tag{4.6}$$

where we used the abbreviations

$$G_{jj}^{nn'}=G_{jj}^{x}(t_n,t_{n'}),\qquad G_{jk}^{nm}=G_{jk}^{x}(t_n,t_{N+m}),\qquad G_{kk}^{mm'}=G_{kk}^{x}(t_{N+m},t_{N+m'}). \tag{4.7}$$

To proceed, it is more convenient to write expression (4.6) as a convolution integral

$$\langle F_1(x(t_1))\cdots F_{N+M}(p(t_{N+M}))\rangle_\Omega^{x_b,x_a}=\prod_{n=1}^{N}\left[\int_{-\infty}^{+\infty}dx_nF_n(x_n)\right]\prod_{m=1}^{M}\left[\int_{-\infty}^{+\infty}\frac{dp_m}{2\pi\hbar}F_{N+m}(p_m)\right]\times\left(\frac{M\Omega}{\hbar}\right)^{(N-M)/2}P(x_1,\ldots,x_N,p_1,\ldots,p_M) \tag{4.8}$$

involving the Gaussian distribution

$$P(x_1,\ldots,p_M)\equiv\frac{1}{(2\pi)^N}\int d^{N+M}v\exp\left\{i\mathbf{w}^T\mathbf{v}-\frac{i}{2}\mathbf{v}^TG\mathbf{v}\right\}. \tag{4.9}$$

The dimensionless vectors $\mathbf{v}$ and $\mathbf{w}$ have $N+M$ components and are defined as

$$\mathbf{v}^T=\left(\left(\frac{\hbar}{M\Omega}\right)^{1/2}\xi_1,\ldots,\left(\frac{\hbar}{M\Omega}\right)^{1/2}\xi_N,\left(\frac{M\Omega}{\hbar}\right)^{1/2}\kappa_1,\ldots,\left(\frac{M\Omega}{\hbar}\right)^{1/2}\kappa_M\right) \tag{4.10}$$

and

$$\mathbf{w}^T=\left(\left(\frac{M\Omega}{\hbar}\right)^{1/2}[x_1-x_{\rm cl}(t_1)],\ldots,\left(\frac{M\Omega}{\hbar}\right)^{1/2}[x_N-x_{\rm cl}(t_N)],-\frac{1}{\sqrt{\hbar M\Omega}}[p_1-p_{\rm cl}(t_{N+1})],\ldots,-\frac{1}{\sqrt{\hbar M\Omega}}[p_M-p_{\rm cl}(t_{N+M})]\right). \tag{4.11}$$

The $(N+M)\times(N+M)$ matrix of Green functions

$$G=\begin{pmatrix}A & B\\ B^T & C\end{pmatrix} \tag{4.12}$$

can be decomposed into block matrices A, B, and C. The $N\times N$ matrix A and the $M\times M$ matrix C are defined by

$$A=\Omega\begin{pmatrix}G_{jj}^{11} & G_{jj}^{12} & \cdots & G_{jj}^{1N}\\ G_{jj}^{12} & G_{jj}^{11} & \cdots & G_{jj}^{2N}\\ \vdots & \vdots & \ddots & \vdots\\ G_{jj}^{1N} & G_{jj}^{2N} & \cdots & G_{jj}^{11}\end{pmatrix},$$

$$C=\frac{1}{\Omega}\begin{pmatrix}G_{kk}^{11} & G_{kk}^{12} & \cdots & G_{kk}^{1M}\\ G_{kk}^{12} & G_{kk}^{11} & \cdots & G_{kk}^{2M}\\ \vdots & \vdots & \ddots & \vdots\\ G_{kk}^{1M} & G_{kk}^{2M} & \cdots & G_{kk}^{11}\end{pmatrix} \tag{4.13}$$

and yield quadratic forms of the position and momentum variables, respectively. The $N\times M$ matrix

$$B=\begin{pmatrix}-G_{jk}^{11} & -G_{jk}^{12} & \cdots & -G_{jk}^{1M}\\ -G_{jk}^{21} & -G_{jk}^{11} & \cdots & -G_{jk}^{2M}\\ \vdots & \vdots & \vdots & \vdots\\ -G_{jk}^{N1} & -G_{jk}^{N2} & \cdots & -G_{jk}^{NM}\end{pmatrix} \tag{4.14}$$

gives rise to quadratic terms which are linear in both position and momentum variables. The multidimensional integral in Eq. (4.9) is of the Fresnel type and can easily be done, yielding an explicit expression for the Gaussian distribution (4.9),

$$P(x_1,\ldots,x_N,p_1,\ldots,p_M)=\frac{1}{\sqrt{i^{N+M}(2\pi)^{N-M}\det G}}\exp\left\{\frac{i}{2}\mathbf{w}^TG^{-1}\mathbf{w}\right\}, \tag{4.15}$$

where G^{-1} represents the matrix inverse of Eq. (4.12) whose block form is

$$G^{-1}=\begin{pmatrix}X^{-1} & -X^{-1}BC^{-1}\\ -C^{-1}B^TX^{-1} & C^{-1}+C^{-1}B^TX^{-1}BC^{-1}\end{pmatrix} \tag{4.16}$$

with the abbreviation

$$X=A-BC^{-1}B^T. \tag{4.17}$$

Since the matrix G may be decomposed as

$$G=\begin{pmatrix}1 & B\\ 0 & C\end{pmatrix}\begin{pmatrix}X & 0\\ C^{-1}B^T & 1\end{pmatrix} \tag{4.18}$$

when the matrix C is regular, the determinant of G factorizes as follows:

$$\det G=\det C\,\det X. \tag{4.19}$$

For singular matrix C but A regular, one may make use of another decomposition,

$$G=\begin{pmatrix}1 & 0\\ B^TA^{-1} & X'\end{pmatrix}\begin{pmatrix}A & B\\ 0 & 1\end{pmatrix}, \tag{4.20}$$

with $X'=C-B^TA^{-1}B$. Then the determinant of G is given by

$$\det G=\det X'\,\det A. \tag{4.21}$$

With the Gaussian distribution (4.15), our result (4.8) constitutes a *smearing formula* which describes the effect of harmonic fluctuations upon arbitrary products of functions of space and momentum variables at different times.

V. GENERALIZED WICK RULES AND FEYNMAN DIAGRAMS

In applications, there often occur correlation functions for mixtures of nonpolynomial functions $F(\tilde{x}(t))$ or $F(\tilde{p}(t))$ and powers according to

$$\begin{aligned}&\langle F(\tilde{x}(t_1))\ \tilde{x}^n(t_2)\rangle_\Omega^{x_b,x_a}7, \quad \langle F(\tilde{x}(t_1))\ \tilde{p}^n(t_2)\rangle_\Omega^{x_b,x_a},\\ &\langle F(\tilde{p}(t_1))\ \tilde{x}^n(t_2)\rangle_\Omega^{x_b,x_a}, \quad \langle F(\tilde{p}(t_1))\ \tilde{p}^n(t_2)\rangle_\Omega^{x_b,x_a}.\end{aligned} \tag{5.1}$$

In order to evaluate such correlation functions, we derive in this section generalized Wick rules and Feynman diagrams on the basis of the smearing formula (4.8).

A. Ordinary Wick rules

It is well known that if one has to calculate expectation values of polynomials with even power, Wick's rule can be written as the sum over all possible permutations of products of two-point functions. We shortly recall this expansion by considering the case of a position-dependent n-point correlation function, n even, defined as

$$G^{(n)}(t_1,\ldots,t_n)=\langle\tilde{x}(t_1)\cdots\tilde{x}(t_n)\rangle_\Omega^{x_b,x_a}. \tag{5.2}$$

Note that it will be sufficient to study only the correlation functions involving the deviations from the classical path, respectively. This expectation value can be decomposed with the help of Wick's expansion,

$$G^{(n)}(t_1,\ldots,t_n)=\sum_{\text{pairs}} G^{(2)}(t_{p(1)},t_{p(2)})\cdots G^{(2)}(t_{p(n-1)},t_{p(n)}), \tag{5.3}$$

where p denotes the operation of pairwise index permutation. Thereby, the Green function $G^{(2)}(t_1,t_2)$ is already given by Eq. (3.46). Note that Eq. (5.3) may be considered as a consequence of a simple derivative rule

$$\langle F(\tilde{x}(t_1))\tilde{x}(t_2)\rangle_\Omega^{x_b,x_a}=\langle\tilde{x}(t_1)\tilde{x}(t_2)\rangle_\Omega^{x_b,x_a}\langle F'(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a} \tag{5.4}$$

with $F'(\tilde{x})=\partial F(\tilde{x})/\partial x$. By applying this recursively, one eventually obtains Eq. (5.3). And conversely, the derivative rule (5.4) can be proved for *polynomial* functions $F(\tilde{x}(t))$, following directly from Wick's theorem (5.3).

The two-point Green functions $G^{(2)}(t_1,t_2)$, occurring in Eq. (5.3), can be considered as a Wick contraction, which we introduce as follows:

$$\underbrace{\tilde{x}(t_1)\,\tilde{x}(t_2)} = \langle\,\tilde{x}(t_1)\,\tilde{x}(t_2)\,\rangle_\Omega^{x_b,x_a} = \frac{i\hbar}{M}G_{jj}(t_1,t_2), \tag{5.5}$$

$$\underbrace{\tilde{x}(t_1)\,\tilde{p}(t_2)} = \langle\,\tilde{x}(t_1)\,\tilde{p}(t_2)\,\rangle_\Omega^{x_b,x_a} = i\hbar G_{jk}(t_1,t_2), \tag{5.6}$$

$$\underbrace{\tilde{p}(t_1)\,\tilde{x}(t_2)} = \langle\,\tilde{p}(t_1)\,\tilde{x}(t_2)\,\rangle_\Omega^{x_b,x_a} = i\hbar G_{kj}(t_1,t_2) = i\hbar G_{jk}(t_2,t_1), \tag{5.7}$$

$$\underbrace{\tilde{p}(t_1)\,\tilde{p}(t_2)} = \langle\,\tilde{p}(t_1)\,\tilde{p}(t_2)\,\rangle_\Omega^{x_b,x_a} = i\hbar M G_{kk}(t_1,t_2). \tag{5.8}$$

Decomposing polynomial correlations of $\tilde{x}(t)$ and $\tilde{p}(t)$ with the help of these contractions corresponding to Eq. (5.3) or successively applying the derivative rule (5.4) leads to the following results:

$$\langle\tilde{x}^n(t_1)\tilde{x}^m(t_2)\rangle_\Omega^{x_b,x_a}=\sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{\min(n,m)} c_l\left[\frac{i\hbar}{M}G_{jj}(t_1,t_1)\right]^{(n-l)/2}\left[\frac{i\hbar}{M}G_{jj}(t_1,t_2)\right]^{l}\left[\frac{i\hbar}{M}G_{jj}(t_2,t_2)\right]^{(m-l)/2}, \tag{5.9}$$

$$\langle\tilde{x}^n(t_1)\tilde{p}^m(t_2)\rangle_\Omega^{x_b,x_a}=\sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{\min(n,m)} c_l\left[\frac{i\hbar}{M}G_{jj}(t_1,t_1)\right]^{(n-l)/2}[i\hbar G_{jk}(t_1,t_2)]^{l}\,[i\hbar M G_{kk}(t_2,t_2)]^{(m-l)/2}, \tag{5.10}$$

$$\langle\tilde{p}^n(t_1)\tilde{x}^m(t_2)\rangle_\Omega^{x_b,x_a}=\sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{\min(n,m)} c_l\,[i\hbar M G_{kk}(t_1,t_1)]^{(n-l)/2}\,[i\hbar G_{jk}(t_2,t_1)]^{l}\left[\frac{i\hbar}{M}G_{jj}(t_2,t_2)\right]^{(m-l)/2}, \tag{5.11}$$

$$\langle\tilde{p}^n(t_1)\tilde{p}^m(t_2)\rangle_\Omega^{x_b,x_a}=\sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{\min(n,m)} c_l\,[i\hbar M G_{kk}(t_1,t_1)]^{(n-l)/2}\,[i\hbar M G_{kk}(t_1,t_2)]^{l}\,[i\hbar M G_{kk}(t_2,t_2)]^{(m-l)/2} \tag{5.12}$$

with the multiplicity factor

$$c_l = \frac{(n-l-1)!!(m-l-1)!!n!m!}{l!(n-l)!(m-l)!}. \tag{5.13}$$

Note that $(-k)!!\equiv 1$ for any positive integer k. For nonvanishing correlation, the sum $n+m$ must be even so that the regulation parameter α is defined as follows:

$$\alpha = \begin{cases} 0, & n,m \text{ even}, \\ 1, & n,m \text{ odd}. \end{cases} \tag{5.14}$$

The contractions defined in Eqs. (5.5)–(5.8) can be used to treat Taylor-expandable functions $F(\tilde{x}(t))$ and $F(\tilde{p}(t))$ only. The desired derivative rules for such correlations read

$$\langle F(\tilde{x}(t_1))\tilde{x}^n(t_2)\rangle_\Omega^{x_b,x_a} = \sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{n} \frac{n!}{(n-l)!!l!}\left[\frac{i\hbar}{M}G_{jj}(t_2,t_2)\right]^{(n-l)/2}\left[\frac{i\hbar}{M}G_{jj}(t_1,t_2)\right]^{l}\langle F^{(l)}(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}, \tag{5.15}$$

$$\langle F(\tilde{x}(t_1))\tilde{p}^n(t_2)\rangle_\Omega^{x_b,x_a} = \sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{n} \frac{n!}{(n-l)!!l!}[i\hbar M G_{kk}(t_2,t_2)]^{(n-l)/2}[i\hbar G_{jk}(t_1,t_2)]^{l}\langle F^{(l)}(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}, \tag{5.16}$$

$$\langle F(\tilde{p}(t_1))\tilde{p}^n(t_2)\rangle_\Omega^{x_b,x_a} = \sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{n} \frac{n!}{(n-l)!!l!}[i\hbar M G_{kk}(t_2,t_2)]^{(n-l)/2}[i\hbar M G_{kk}(t_1,t_2)]^{l}\langle F^{(l)}(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a}, \tag{5.17}$$

$$\langle F(\tilde{p}(t_1))\tilde{x}^n(t_2)\rangle_\Omega^{x_b,x_a} = \sum_{l=\alpha,\alpha+2,\alpha+4,\ldots}^{n} \frac{n!}{(n-l)!!l!}\left[\frac{i\hbar}{M}G_{jj}(t_2,t_2)\right]^{(n-l)/2}[i\hbar G_{jk}(t_2,t_1)]^{l}\langle F^{(l)}(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a}. \tag{5.18}$$

The parameter α distinguishes between even and odd power n:

$$\alpha = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}, \end{cases} \tag{5.19}$$

since even (odd) powers of n lead to even (odd) derivatives of the function $F(\tilde{x}(t_1))$. The lth derivative $F^{(l)}(\tilde{x}(t_1))$ is formed with respect to $x(t_1)$, and $F^{(l)}(\tilde{p}(t_1))$ is the lth derivative with respect to $p(t_1)$. Note that in the last line the Green function G_{jk} appears with exchanged time arguments, which in this case happens to be inessential due to the symmetry $G_{jk}(t_2,t_1) = G_{kj}(t_1,t_2)$.

B. Generalized Wick rule

According to their derivation, the contractions (5.15)–(5.18) are only applicable to functions $F(\tilde{x}(t))$ and $F(\tilde{p}(t))$ which can be Taylor-expanded. In the following, we will show with the help of the smearing formula (4.8) that these derivative rules remain valid for functions $F(\tilde{x}(t))$ and $F(\tilde{p}(t))$ with Laurent expansions. Expectations of this type appear in variational perturbation theory (see for position-position coupling Ref. [7]). Since the proceeding is similar in all the cases (5.15)–(5.18), we shall only discuss the expectation value

$$\langle F(\tilde{x}(t_1))\tilde{p}^n(t_2)\rangle_\Omega^{x_b,x_a} \tag{5.20}$$

in detail. For this we consider the generating functional of all such expectation values following from Eq. (4.8),

$$\langle F(\tilde{x}(t_1))e^{j\tilde{p}^n(t_2)}\rangle_\Omega^{x_b,x_a} = \frac{1}{\sqrt{-\det G}}\int_{-\infty}^{+\infty}dx\,F(x)\int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}e^{jp} \times\exp\left\{\frac{i}{2\det G}\left[\frac{M}{\hbar}G_{kk}(t_2,t_2)x^2 - 2\frac{1}{\hbar}G_{jk}(t_1,t_2)xp + \frac{1}{\hbar M}G_{jj}(t_1,t_1)p^2\right]\right\}. \tag{5.21}$$

The p integration can easily be done, leading to

$$\langle F(\tilde{x}(t_1))e^{j\tilde{p}^n(t_2)}\rangle_\Omega^{x_b,x_a} = e^{i\hbar M G_{kk}(t_2,t_2)j^2/2}\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{2\pi i\hbar G_{jj}(t_1,t_2)/M}}F[x+i\hbar G_{jk}(t_1,t_2)j]e^{iMx^2/2\hbar G_{jj}(t_1,t_1)}$$
$$= e^{i\hbar M G_{kk}(t_2,t_2)j^2/2}\sum_{l=0}^{\infty}\frac{1}{l!}[i\hbar G_{jk}(t_1,t_2)j]^l\langle F^{(l)}(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}. \tag{5.22}$$

The correlation of two functions at different times has been reduced to a single-time expectation value of the lth derivative of the function $F(\tilde{x}(t_1))$ with respect to $x(t_1)$, denoted by $F^{(l)}(\tilde{x}(t_1))$, with Green functions describing the dependence on the second time. Expanding both sides in powers of j, we reobtain Eq. (5.16).

Now we demonstrate that the derivative rules (5.15)–(5.18) for Laurent-expandable functions $F(\tilde{x}(t))$ and $F(\tilde{p}(t))$ also follow from generalized Wick rules. Without restriction of universality, we only consider the expectation value

$$\langle F(\tilde{x}(t_1))\tilde{x}^n(t_2)\rangle_\Omega^{x_b,x_a}. \tag{5.23}$$

The proceeding to reduce the power of the polynomial at the expense of the function $F(\tilde{x}(t_1))$ is as follows.

(1a) If possible ($n \geqslant 2$), contract $\tilde{x}(t_2)\tilde{x}(t_2)$ with multiplicity $(n-1)$, giving

$$(n-1)\,\underbracket{\tilde{x}(t_2)\,\tilde{x}(t_2)}\,\langle F(\tilde{x}(t_1))\,\tilde{x}^{n-2}(t_2)\rangle_\Omega^{x_b,x_a}, \tag{5.24}$$

or else jump to (1b) directly.

(1b) Contract $F(\tilde{x}(t_1))\tilde{x}(t_2)$ and let the remaining polynomial invariant. We define this contraction by the symbol

$$\underbracket{F(\tilde{x}(t_1))\,\tilde{x}(t_2)}\,\tilde{x}^{n-1}(t_2) = \underbracket{\tilde{x}(t_1)\,\tilde{x}(t_2)}\,\langle F'(\tilde{x}(t_1))\,\tilde{x}^{n-1}(t_2)\rangle_\Omega^{x_b,x_a}. \tag{5.25}$$

(1c) Add the terms (1a) and (1b).

(2) Repeat steps (1a)–(1c) until only expectation values of $F(\tilde{x})$ or expectations of its derivatives remain.

Summarizing, we can express the first power reduction by the generalized Wick rule ($n \geqslant 2$),

$$\langle F(\tilde{x}(t_1))\,\tilde{x}^n(t_2)\rangle_\Omega^{x_b,x_a} = (n-1)\,\underbracket{\tilde{x}(t_2)\,\tilde{x}(t_2)}\,\langle F(\tilde{x}(t_1))\,\tilde{x}^{n-2}(t_2)\rangle_\Omega^{x_b,x_a} + \underbracket{F(\tilde{x}(t_1))\,\tilde{x}(t_2)}\,\tilde{x}^{n-1}(t_2) \tag{5.26}$$

with the contraction rules defined in Eqs. (5.5) and (5.25). For $n=1$, we obtain

$$\langle F(\tilde{x}(t_1))\,\tilde{x}(t_2)\rangle_\Omega^{x_b,x_a} = \underbracket{\tilde{x}(t_1)\,\tilde{x}(t_2)}\,\langle F'(x(t_1))\rangle_\Omega^{x_b,x_a}, \tag{5.27}$$

which is valid for *any* function $F(\tilde{x}(t))$ generalizing the rule (5.4) that was proved for polynomial functions only. Recursively applying this power reduction, we finally end up with the derivative rule (5.15). Note that the generalization of Wick's rule for mixed position momentum or pure momentum couplings is done along similar lines, leading to the derivative rules (5.16)–(5.18).

C. New Feynman-like rules for nonpolynomial interactions

Higher-order perturbation expressions become usually complicated. For simple polynomial interactions, Feynman diagrams are a useful tool to classify perturbative contributions with the help of graphical rules. Here, we are going to set up analogous diagrammatic rules for perturbation expansions for nonpolynomial interactions $V(x(t),p(t))$, whose contributions may be expressed as expectations values

$$\int_{t_a}^{t_b}dt_n\cdots\int_{t_a}^{t_b}dt_1\langle V(x(t_n),p(t_n))\cdots V(x(t_1),p(t_1))\rangle_\Omega^{x_b,x_a}. \tag{5.28}$$

From Eqs. (5.5)–(5.8) it follows that we have four basic propagators whose graphical representation may be defined as (setting $\hbar = M = 1$ from now on)

t_1 ——— t_2 $\equiv \langle \tilde{x}(t_1)\,\tilde{x}(t_2)\rangle_\Omega^{x_b,x_a} = iG_{jj}(t_1,t_2)$,

t_1 ～～～ t_2 $\equiv \langle \tilde{p}(t_1)\,\tilde{p}(t_2)\rangle_\Omega^{x_b,x_a} = iG_{kk}(t_1,t_2)$,

t_1 --◄--- t_2 $\equiv \langle \tilde{x}(t_1)\,\tilde{p}(t_2)\rangle_\Omega^{x_b,x_a} = iG_{jk}(t_1,t_2)$,

t_1 ---►-- t_2 $\equiv \langle \tilde{p}(t_1)\,\tilde{x}(t_2)\rangle_\Omega^{x_b,x_a} = iG_{kj}(t_1,t_2) = iG_{jk}(t_2,t_1)$.

A vertex is represented as usual by a small dot. The time variable is integrated over at a vertex in a perturbation expansion,

$$\bullet \equiv \int_{t_a}^{t_b}dt.$$

We now introduce the diagrammatic representations of the expectation value of arbitrary functions $F(\tilde{x}(t))$ or $F(\tilde{p}(t))$ and their derivatives as

★ $\equiv \int_{t_a}^{t_b} dt \, \langle F(\tilde{x}(t)) \rangle_{\Omega}^{x_b,x_a}$, ☆ $\equiv \int_{t_a}^{t_b} dt \, \langle F(\tilde{p}(t)) \rangle_{\Omega}^{x_b,x_a}$,

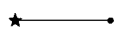
 $\equiv \int_{t_a}^{t_b} dt \, \langle F'(\tilde{x}(t)) \rangle_{\Omega}^{x_b,x_a}$,  $\equiv \int_{t_a}^{t_b} dt \, \langle F'(\tilde{p}(t)) \rangle_{\Omega}^{x_b,x_a}$,

 $\equiv \int_{t_a}^{t_b} dt \, \langle F''(\tilde{x}(t)) \rangle_{\Omega}^{x_b,x_a}$,  $\equiv \int_{t_a}^{t_b} dt \, \langle F''(\tilde{p}(t)) \rangle_{\Omega}^{x_b,x_a}$,

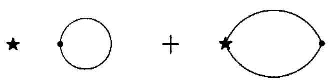
⋮ ⋮ .

With these elements, we can compose Feynman graphs for two-point correlation functions of the type (5.1) for arbitrary n by successively applying the generalized Wick rule (5.26) or directly using the derivative relations (5.15)–(5.18). The general results become obvious by giving explicitly a graphical representation of the following four correlation functions:

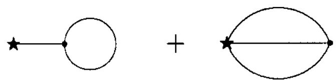
$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{x}(t_1))\tilde{x}(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \; iG_{jj}(t_1,t_2)\langle F'(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a}$$

$\equiv$  , (5.29)

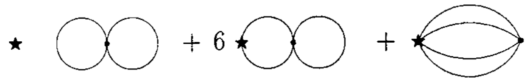
$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{x}(t_1))\tilde{x}^2(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \; \{iG_{jj}(t_2,t_2)\langle F(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{jj}(t_1,t_2)]^2 \langle F''(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$

$\equiv$  , (5.30)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{x}(t_1))\tilde{x}^3(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \; \{3iG_{jj}(t_1,t_2) iG_{jj}(t_2,t_2)\langle F'(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a}$$
$$+ [iG_{jj}(t_1,t_2)]^3 \langle F'''(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$

$\equiv$  , (5.31)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{x}(t_1))\tilde{x}^4(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \; \{[iG_{jj}(t_2,t_2)]^2 \langle F(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a} + 6[iG_{jj}(t_1,t_2)]^2 iG_{jj}(t_2,t_2)$$
$$\times \langle F''(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{jj}(t_1,t_2)]^4 \langle F^{(4)}(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$

$\equiv$  . (5.32)

Mixed position-momentum and momentum-momentum correlations and their graphical representations are given in the Appendix.

The consideration of higher-order correlations with more than one function $F(\tilde{x}(t))$ or $F(\tilde{p}(t))$ can be reduced to the results (5.9)–(5.12) or (5.15)–(5.18) by expanding them with respect to the classical path or momentum, respectively. By expanding both functions in the expectation value, one obtains, for example,

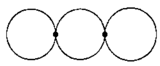
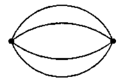
$$\langle F_1(\tilde{x}(t_1)) F_2(\tilde{x}(t_2))\rangle_{\Omega}^{x_b,x_a} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} f_{1,m} f_{2,n} \langle \tilde{x}^m(t_1)\tilde{x}^n(t_2)\rangle_{\Omega}^{x_b,x_a} \tag{5.33}$$

with

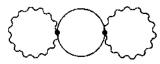
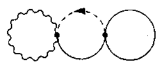
$$f_{i,m} = F^{(m)}(0), \quad i=1,2. \tag{5.34}$$

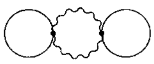
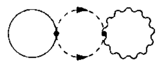
But constructing graphical rules for such general correlations is more involved due to the various summations over products of powers of propagators $G_{jj}(t_i,t_j)$ with $i,j=1,2$.

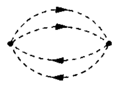
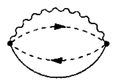
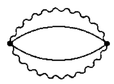
Finally, we apply the diagrammatic rules to the anharmonic oscillator with $\tilde{x}^4$ interaction, which is a powerful system being discussed in detail with the help of a perturbation expansion ([1], Chap. 3). With the Green functions given by Eqs. (3.22), (3.44), and (3.45), the two-point correlation for the anharmonic system with arbitrary time-dependent frequency can then be expressed graphically, yielding the known decomposition for the second-order perturbative contribution

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle \, \tilde{x}^4(t_1) \, \tilde{x}^4(t_2) \, \rangle_{\Omega,c}^{x_b,x_a} \equiv 72$$  $+ 24$  (5.35)

with subscript c indicating that we restrict to connected graphs only. Beyond this, our theory allows us to describe nonstandard systems with polynomial interactions (5.28) depending on both position and momentum, to higher order. Finally, we want to give the graphs for a four-interaction $\tilde{x}^2\tilde{p}^2$ to the second order to see the variations of possible graphs in comparison with Eq. (5.35):

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle \, \tilde{x}^2(t_1) \, \tilde{p}^2(t_1) \, \tilde{x}^2(t_2) \, \tilde{p}^2(t_2) \, \rangle_{\Omega,c}^{x_b,x_a} \equiv 2$$  $+ 16$ 

$+16$  $+ 2$  $+ 4$  $+ 16$ 

$+16$  $+ 4$  $+ 16$  $+ 4$  . (5.36)

We see that we have the same class of graphs already occurring in Eq. (5.35), however, with different propagators connecting the vertices. Thus, both classes decay into subclasses with different multiplicities, but the total numbers remain 72 and 24 for each type of class, respectively. Furthermore, all graphs are vacuumlike graphs. Eventually, it is easy to construct the Feynman graphs for polynomial correlations higher than second order by applying Wick's rule or the Feynman rules given in this section.

VI. SIMPLIFICATIONS FOR PERIODIC PATHS

Up to now, we discussed the harmonic time evolution amplitude with arbitrary frequency and external sources $k(t), i(t)$ and corresponding Green functions fulfilling Dirichlet boundary conditions. In the sense of the quantum-mechanical partition function

$$Z=\int_{-\infty}^{+\infty} dx \; (x t_b | x t_a), \tag{6.1}$$

which is an integral over the time evolution amplitude for closed paths, it is of interest to investigate the generating functional for closed paths. In analogy to Eq. (6.1), we define

$$Z[k(t),i(t)]=\int_{-\infty}^{+\infty} dx \; (x t_b | x t_a)[k(t),i(t)] \tag{6.2}$$

with Eq. (3.42) for $x_a=x_b=x$. One immediately observes that $Z=Z[0,0]$. The integral is easily done, giving

$$Z[k(t),i(t)]=\frac{1}{\sqrt{\dot{D}_a(t_b)-\dot{D}_b(t_a)-2}} \times \exp\left\{ -\frac{i}{2\hbar}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \times \left[\frac{1}{M} j(t)\tilde{G}_{jj}^x(t,t')j(t') + j(t)\tilde{G}_{jk}^x(t,t')k(t') + k(t)\tilde{G}_{kj}^x(t,t')j(t') + Mk(t)\tilde{G}_{kk}^x(t,t')k(t') \right] \right\}. \tag{6.3}$$

The Green functions, expressed with fundamental solutions (3.2) and (3.3), are found to be

$$\tilde{G}_{jj}^x(t,t')=\frac{1}{D_a(t_b)}\left[G_{jj}^x(t,t')+\frac{1}{a(t_a,t_b)} g(t)g(t') \right], \tag{6.4}$$

$$\tilde{G}_{jk}^x(t,t')=\frac{1}{D_a(t_b)}\left[G_{jk}^x(t,t')+\frac{1}{a(t_a,t_b)} g(t)\dot{g}(t') \right] = \tilde{G}_{kj}^x(t',t), \tag{6.5}$$

$$\tilde{G}^x_{kk}(t,t') = \frac{1}{D_a(t_b)}\left[G^x_{kk}(t,t') + \frac{1}{a(t_a,t_b)} \dot{g}(t)\dot{g}(t') \right], \tag{6.6}$$

with

$$a(t,t') = \dot{D}_a(t') - \dot{D}_b(t) - 2. \tag{6.7}$$

Since the function

$$g(t) = D_a(t) + D_b(t) \tag{6.8}$$

is periodic, $g(t_a) = g(t_b)$, due to conditions (3.2), (3.3), and (3.6), also the Green function $\tilde{G}^x_{jj}(t,t')$ becomes periodic,

$$\tilde{G}^x_{jj}(t_a,t') = \tilde{G}^x_{jj}(t_b,t'). \tag{6.9}$$

In analogy to the harmonic propagator without external sources (4.1), we can define expectation values consisting of N position-dependent functions and M momentum-dependent functions by

$$\begin{aligned} &\langle F_1(x(t_1))F_2(x(t_2))\cdots F_{N+M}(p(t_{N+M}))\rangle_\Omega \\ &= \frac{1}{Z} \oint \frac{\mathcal{D}x\mathcal{D}p}{2\pi\hbar} F_1(x(t_1))F_2(x(t_2))\cdots F_{N+M}(p(t_{N+M})) \\ &\quad \times \exp\left\{ \frac{i}{\hbar}\mathcal{A}[p,x;0,0] \right\}. \end{aligned} \tag{6.10}$$

We remark that the generalization of Wick's rule and the graphical representation with the help of Feynman diagrams of such correlation functions is exactly the same as given in the preceding section after substituting the Green functions $G(t,t')$ by $\tilde{G}(t,t')$ and expectation values (4.1) by Eq. (6.10).

VII. SUMMARY AND OUTLOOK

We have reduced generating functionals with fixed end points to those with vanishing end points by adding special singular sources to the currents. The new generating functionals were calculated explicitly for the harmonic oscillator with time-dependent frequency. From this expression, a smearing formula was derived which serves to calculate correlation functions for arbitrary polynomial or nonpolynomial position- and momentum-dependent couplings. We have further found a generalization of Wick's theorem of decomposing correlation functions involving functions of the canonic variables of the system. This gives rise to certain generalized Feynman rules for position- and momentum-dependent expectation values.

Due to its universality, the theory should serve as a basis for investigating physical systems with a nonstandard Hamiltonian via perturbation theory and its variational extension. Note that a perturbation theory for momentum-dependent interactions arises in important field theories such as the nonlinear σ model. Our work is supposed to lay the foundation for a more efficient perturbation treatment of such a theory.

ACKNOWLEDGMENT

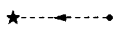
One of us (M.B.) is supported by the Studienstiftung des deutschen Volkes.

APPENDIX: GENERALIZED CORRELATION FUNCTIONS

In this appendix we give the expectations for the correlation between a general position- or momentum-dependent function and a polynomial up to order $n=4$.

Position-momentum coupling

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{x}(t_1))\tilde{p}(t_2)\rangle_\Omega^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, iG_{jk}(t_1,t_2)\langle F'(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}$$

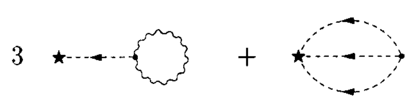
$\equiv$  , (A1)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{x}(t_1))\tilde{p}^2(t_2)\rangle_\Omega^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{ iG_{kk}(t_2,t_2)\langle F(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a} + [iG_{jk}(t_1,t_2)]^2\langle F''(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}\}$$

$\equiv$  , (A2)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{x}(t_1))\tilde{p}^3(t_2)\rangle_\Omega^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{ 3iG_{jk}(t_1,t_2)iG_{kk}(t_2,t_2)\langle F'(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}$$

$$+ [iG_{jk}(t_1,t_2)]^3\langle F'''(\tilde{x}(t_1))\rangle_\Omega^{x_b,x_a}\}$$

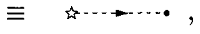
$\equiv$  , (A3)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{x}(t_1))\tilde{p}^4(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{[iG_{kk}(t_2,t_2)]^2 \langle F(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a} + 6[iG_{jk}(t_1,t_2)]^2 iG_{kk}(t_2,t_2) \times \langle F''(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{jk}(t_1,t_2)]^4 \langle F^{(4)}(\tilde{x}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$

(A4)

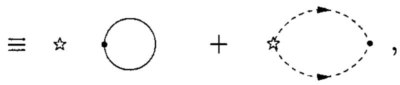
Momentum-position coupling

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{x}(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, iG_{kj}(t_1,t_2) \langle F'(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a}$$



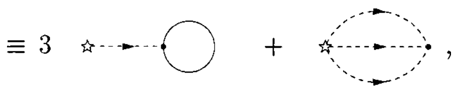
(A5)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{x}^2(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{iG_{jj}(t_2,t_2) \langle F(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{kj}(t_1,t_2)]^2 \langle F''(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$



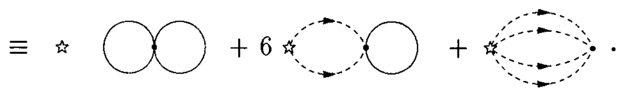
(A6)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{x}^3(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{3iG_{kj}(t_1,t_2) iG_{jj}(t_2,t_2) \langle F'(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{kj}(t_1,t_2)]^3 \langle F'''(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$



(A7)

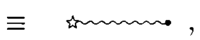
$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{p}(t_1))\tilde{x}^4(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \{[iG_{jj}(t_2,t_2)]^2 \langle F(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a} + 6[iG_{kj}(t_1,t_2)]^2 iG_{jj}(t_2,t_2) \times \langle F''(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{kj}(t_1,t_2)]^4 \langle F^{(4)}(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$



(A8)

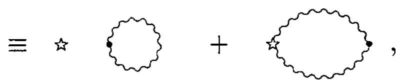
Momentum-momentum coupling

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{p}(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, iG_{kk}(t_1,t_2) \langle F'(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a}$$



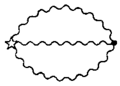
(A9)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{p}^2(t_2)\rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{iG_{kk}(t_2,t_2) \langle F(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a} + [iG_{kk}(t_1,t_2)]^2 \langle F''(\tilde{p}(t_1))\rangle_{\Omega}^{x_b,x_a}\}$$

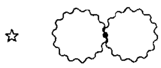
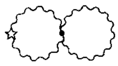
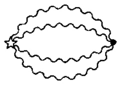


(A10)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{p}^3(t_2)\rangle_\Omega^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{3iG_{kk}(t_1,t_2)iG_{kk}(t_2,t_2)\langle F'(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a} + [iG_{kk}(t_1,t_2)]^3 \langle F'''(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a}\}$$

$\equiv 3$  $+$  , (A11)

$$\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \langle F(\tilde{p}(t_1))\tilde{p}^4(t_2)\rangle_\Omega^{x_b,x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \, \{[iG_{kk}(t_2,t_2)]^2 \langle F(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a} + 6[iG_{kk}(t_1,t_2)]^2 iG_{kk}(t_2,t_2) \times \langle F''(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a} + [iG_{kk}(t_1,t_2)]^4 \langle F^{(4)}(\tilde{p}(t_1))\rangle_\Omega^{x_b,x_a}\}$$

$\equiv$  $+ 6$  $+$  . (A12)

The case of position-position coupling has already been calculated in Sec. V C.

[1] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995).
[2] See Chap. 2 in the forthcoming third edition of the textbook of Ref. [1] (http://www.physik.fu-berlin.de/kleinert/kleiner_reb3/3rded.html).
[3] H. Kleinert and A. Chervyakov, e-print physics/9712048.
[4] H. Kleinert and A. Chervyakov, Phys. Lett. A **245**, 345 (1998).
[5] I. M. Gelfand and A. M. Yaglom, J. Math. Phys. **1**, 48 (1960).
[6] H. Kleinert, W. Kürzinger, and A. Pelster, J. Phys. A **31**, 8307 (1998).
[7] M. Bachmann, H. Kleinert, and A. Pelster, e-print quant-ph/9812063.