Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

FIVE-LOOP EXPANSION OF THE ϕ^4-THEORY AND CRITICAL EXPONENTS FROM STRONG-COUPLING THEORY

V. SCHULTE-FROHLINDE

Lyman Laboratory, Department of Physics, Harvard University,
12 Oxford Street Cambridge, MA 02138, USA
E-mail: frohlind@cmts.harvard.edu

The full analytic reevaluation of all diagrams up to five loops of the $O(N)$-symmetric ϕ^4-theory led to the correction of the ε-expansions of the β-function and the anomalous dimensions. These expansions were ideal testing grounds for various resummation techniques. Especially Hagen Kleinert's strong-coupling approach led to a reformulation of the renormalization group theory in terms of the bare parameters.

1 Introduction

The scalar quantum field theory with ϕ^4-interaction correctly describes many experimentally observable features of critical phenomena. Field theoretic renormalization group techniques [1] in $D = 4 - \varepsilon$ dimensions [2–4] combined with Borel resummation methods of the resulting ε-expansions [5] lead to extremely accurate determinations of the critical exponents of all $O(N)$ universality classes. The renormalization group (RG) functions of the ϕ^4-theory were first calculated analytically close to four dimensions using dimensional regularization [6] and the minimal subtraction (MS) scheme [7] in three- and four-loop approximations [8,9]. This calculation was extended to the five-loop level [10–12] after the ingenious invention of special reduction algorithms for the integrals [13,14]. The critical exponents were obtained as ε-expansions [3] up to the order ε^5. These expansions have to be evaluated for $\varepsilon = 1$ in order to obtain results in three dimensions.

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

When the analytic five-loop calculation of the β-function and the anomalous dimensions was completed in 1983/1984, Hagen Kleinert had the idea to use the new algorithms to automatize the calculation of Feynman diagrams and their ε-expansions In 1989, this idea then became a thesis project for my colleague Joachim Neu and me. Our first step in this rather lengthy project was an independent recalculation of the five-loop perturbation series using the same techniques [10,13]. Unfortunately, we could not reproduce the results for some of the diagrams. Hagen Kleinert sent us to Moscow to discuss our results, a trip which led to the discovery of errors in six of the 135 diagrams and to our first publication [15]. In the subsequent years, the perturbation expansions for the critical exponents were used to study old and new resummation methods leading among other results [16] to Kleinert's strong-coupling approach to the renormalization group [17,18].

Here, we will summarize the five-loop calculations [15] and then present the strong-coupling approach to resum the ε-expansion of the critical exponents [19]. Details can be found in our textbook [20].

2 Five-Loop Expansion of the ϕ^4-Theory

We consider the $O(N)$-symmetric theory of N-dimensional real scalar fields ϕ_B with the Lagrangian

$$L_B(x) = \frac{1}{2}\left[\partial\phi_B(x)\right]^2 + \frac{1}{2}m_B^2\phi_B^2(x) + (4\pi)^2\frac{\lambda_B}{4!}\left[\phi_B^2(x)\right]^2 , \qquad (1)$$

in Euclidean space with $D = 4 - \varepsilon$ dimensions. The bare (unrenormalized) coupling constant $\lambda_{\rm B}$ and mass $m_{\rm B}$ are expressed via renormalized ones as

$$\lambda_{\rm B} = \mu^\varepsilon Z_g g = \mu^\varepsilon \frac{Z_4}{(Z_2)^2} g, \qquad m_{\rm B}^2 = Z_{m^2} m^2 = \frac{Z_{\phi^2}}{Z_2} m^2 \ . \qquad (2)$$

Here μ is the unit of mass in dimensional regularization and Z_4, Z_2, Z_{m^2}, Z_g are the renormalization constants of the vertex function, propagator, mass, and coupling constant, respectively, with Z_{ϕ^2} being the renormalization constant of the two-point function obtained from the propagator by the insertion of the vertex $(-\phi^2)$ in all possible ways [9]. In the MS-scheme the renormalization constants do not depend on dimensional parameters and are expressible as series in $1/\varepsilon$ with purely g-dependent coefficients:

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{Z_{i,k}(g)}{\varepsilon^k} \ , \qquad (3)$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

where $i = g, m^2, 2, 4, \phi^2$. The β-function and the anomalous dimensions entering the RG equations are expressed in the standard way as follows:

$$\beta(g) = \left.\frac{d\, g}{d\, \ln \mu}\right|_{\lambda_{\rm B}} = -\varepsilon\, g + g \frac{\partial\, Z_{g,1}}{\partial\, g} \ , \tag{4}$$

$$\gamma_m = \left.\frac{d\, \ln m}{d\, \ln \mu}\right|_{\lambda_{\rm B}} = -\frac{d\, \ln Z_{m^2}}{d\, \ln \mu^2} = \frac{1}{2} g \frac{\partial\, Z_{m^2,1}}{\partial\, g} \ , \tag{5}$$

$$\gamma_i(g) = \left.\frac{d\, \ln Z_i}{d\, \ln \mu^2}\right|_{\lambda_{\rm B}} = -\frac{1}{2} g \frac{\partial\, Z_{i,1}}{\partial\, g} \ , \quad i = 2, 4, \phi^2 \ . \tag{6}$$

To determine all RG functions up to five loops we calculated the five-loop approximation to the three constants Z_2, Z_4, and Z_{ϕ^2}. The constant Z_2 contains the counterterms of the 11 five-loop propagator diagrams. The constant Z_4 receives contributions from 124 vertex diagrams. Of these diagrams, 90 contribute to Z_{ϕ^2} after appropriate changes of combinatorial factors.

We have used the same methods as in the previous works [10,13] to calculate the counterterms from the dimensionally regularized Feynman integrals, namely, the method of infrared rearrangement [21], the Gegenbauer polynomial x-space technique (GPXT) [14], the integration-by-parts algorithm [22], and the R- and R^*-operations [23]. These methods allow to proceed with the calculation of massless integrals with only one external momentum. The renormalization is carried out recursively and for each Feynman diagram separately. The higher-order diagrams are then algebraically reduced to one-loop integrations by the integration-by-parts algorithms. Restrictions of the applicability of these algorithms have so far prevented the complete automatization on a computer.

Some of the diagrams do not follow the general scheme. Three diagrams were calculated analytically first [11] by using the so-called method of uniqueness, later the same results were obtained by using the Gegenbauer polynomials in x-space together with several non-trivial tricks [24]. A detailed description of the calculations including the diagramwise results is presented elsewhere [20].

The analytic results of the five-loop approximations to the RG functions $\beta(g)$, $\gamma_2(g)$ and $\gamma_m(g)$ are expansions in g with N-dependent coefficients. The number ε appears only once in the β-function. These RG functions can now be used to calculate the ε-expansions of the critical exponents which describe

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

the behavior of a statistical system near the critical point of the second-order phase transition [4]. Close to the critical temperature $T = T_{\rm C}$, the asymptotic behavior of the correlation function for $|\mathbf{x}| \to \infty$ has the form

$$\Gamma(\mathbf{x}) \sim \frac{e^{-|\mathbf{x}|/\xi}}{|\mathbf{x}|^{D-2+\eta}} \ . \tag{7}$$

Close to $T_{\rm C}$, the correlation length ξ behaves for $\tau = T - T_{\rm C} \to 0$ as

$$\xi \sim \tau^{-\nu}(1 + \text{const} \cdot \tau^{\omega\nu} + \ldots) \ . \tag{8}$$

The three critical exponents η, ν, and ω defined in this way completely specify the critical behavior of the system.

The behavior (7) and (8) is found for the ϕ^4-theory if $\mu \to 0$ as $T \to T_C$. In this limit the coupling constant g approaches the so-called infrared-stable fixed point which is determined by the condition

$$\beta(g^*) = 0, \quad \beta'(g^*) = \left[\partial\beta(g)/\partial g\right]_{g=g^*} > 0 \ . \tag{9}$$

The fixed point g^* is determined as an expansion in ε:

$$g^* = \sum_{k=1}^{\infty} g^{(k)} \varepsilon^k \ . \tag{10}$$

Approaching the fixed point, the renormalized mass goes to zero such that $\xi = 1/m$ behaves like (8). The resulting formulas for the critical exponents are

$$\eta = 2\gamma_2(g^*) \ , \quad 1/\nu = 2[1 - \gamma_m(g^*)] \ , \quad \omega = \beta'(g^*) \ , \tag{11}$$

each emerging as an ε-expansion up to order ε^5 [15,20].

It is known that the ε-expansions are asymptotic series, and special resummation techniques [5,25] should be applied to obtain reliable estimates of the critical exponents. One such technique will be described now.

3 Strong-Coupling Theory

In 1998, Hagen Kleinert has developed a new approach [26,27] to critical exponents of field theories based on the strong-coupling limit of variational perturbation expansions [28,29]. This limit is relevant for critical phenomena if the renormalization constants are expressed in terms of the unrenormalized

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

coupling constant since the infrared-stable fixed point is approached for infinite g_B: $g(g_B) \to g^*$ for $g_B \to \infty$. This idea has been applied successfully to $O(N)$-symmetric ϕ^4-theories in three and $4-\varepsilon$ dimensions [17–19], yielding the three fundamental critical exponents ν, η, ω with high accuracy.

From model studies of perturbation expansions of the anharmonic oscillator it is known that variational perturbation expansions possess good strong-coupling limits [30,31], with a speed of convergence governed by the convergence radius of the strong-coupling expansion [32,33]. This has enabled Hagen Kleinert to set up an algorithm [29] for deriving uniformly convergent approximations to functions of which one knows a few initial Taylor coefficients and an important scaling property: the functions approach a constant value with a given inverse power of the variable. The renormalized coupling constant g and the critical exponents of a ϕ^4-theory have precisely this property as a function of the bare coupling constant g_B. In $D = 4-\varepsilon$ dimensions the approach is parameterized as follows [26]

$$g(g_B) = g^* - \frac{\text{const}}{g_B^{\omega/\varepsilon}} + \dots \ , \tag{12}$$

where g^* is the infrared-stable fixed point, and ω is called the critical exponent of the approach to scaling [compare Eqs. (8) and (11)]. This exponent is universal, governing the approach to scaling of every function $F(g)$,

$$f(g_B) = F(g(g_B)) = F(g^*) + F'(g^*) \times \frac{\text{const}}{g_B} \equiv f^* + \frac{\text{const}'}{g_B^{\omega/\varepsilon}}. \tag{13}$$

Strong-coupling theory is designed to calculate f^* and ω. Let $f(g_B)$ be a function with this behavior and suppose that we know its first $L+1$ expansion terms,

$$f_L(g_B) = \sum_{l=0}^{L} a_l g_B^l. \tag{14}$$

More specifically than in Eq. (12), we assume that $f(g_B)$ approaches its constant strong-coupling limit f^* in the form of an inverse power series

$$f_M(g_B) = \sum_{m=0}^{M} b_m (g_B^{-2/q})^m, \tag{15}$$

with a finite radius of convergence [34]. Then the Lth approximation to the

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann
(World Scientific, Singapore, 2001).

value f^* is obtained from the strong-coupling formula [17,26,27]

$$f_L^* = \operatorname*{opt}_{\hat{g}_B}\left[\sum_{l=0}^{L} a_l v_l \hat{g}_B^l\right]. \tag{16}$$

The quantities

$$v_l \equiv \sum_{k=0}^{L-l} \binom{-ql/2}{k} (-1)^k \tag{17}$$

are simply binomial expansions of $(1-1)^{-ql/2}$ up to the order $L-l$. The expression in brackets in Eq. (16) has to be optimized in the variational parameter $\hat{g}_B$. The optimum is the smoothest among all real extrema. If there are no real extrema, the turning points serve the same purpose.

3.1 Application to Renormalization Constants and Critical Exponents

Going back to Eqs. (1) and (2) we now set the scale parameter μ equal to the physical mass m and consider all quantities as functions of $g_B = \lambda_B/m^\varepsilon$. Now, instead of μ, we let m_B^2 go to zero like $\tau = \text{const}\times(T-T_c)$ as the temperature T approaches the critical temperature T_c, and assume that also m^2 goes to zero, and thus g_B to infinity. The latter assumption turns out to be self-consistent. Assuming the theory to scale as suggested by experiments, we now determine the value of the renormalized coupling constant g in the strong-coupling limit $g_B \to \infty$, and also of the exponent ω, assuming the behavior (12). First we apply formula (16) to the logarithmic derivative $s(g_B)$ of the function $g(g_B)$:

$$s(g_B) \equiv g_B g'(g_B)/g(g_B). \tag{18}$$

Setting $s_L^* = 0$ determines the approximation ω_L to ω.

The other critical exponents are found as follows. If we assume that the ratios m^2/m_B^2 and ϕ^2/ϕ_B^2 have a limiting power-law behavior for small m

$$\frac{m^2}{m_B^2} \propto g_B^{-\eta_m/\varepsilon} \propto m^{\eta_m}, \qquad \frac{\phi^2}{\phi_B^2} \propto g_B^{\eta/\varepsilon} \propto m^{-\eta}, \tag{19}$$

the powers η_m and η can be calculated from the strong-coupling limits of the logarithmic derivatives

$$\eta_m(g_B) = -\varepsilon \frac{d}{d\log g_B} \log\frac{m^2}{m_B^2}, \qquad \eta(g_B) = \varepsilon \frac{d}{d\log g_B} \log\frac{\phi^2}{\phi_B^2}. \tag{20}$$

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

When approaching the second-order phase transition, where the bare mass m_B^2 vanishes like $\tau \equiv (T - T_c)$, the physical mass m^2 vanishes with a different power of τ. This power is obtained from the first equation in (19), which shows that $m \propto \tau^{1/(2-\eta_m)}$. In experiments one observes that the correlation length of fluctuations $\xi = 1/m$ increases near T_c like $\tau^{-\nu}$. A comparison with the previous equation shows that the critical exponent ν is equal to $1/(2-\eta_m)$. Similarly we see from the second equation in (19) that the scaling dimension $D/2 - 1$ of the free field ϕ_B for $T \to T_c$ is changed in the strong-coupling limit to $D/2 - 1 + \eta/2$, the number η being the anomalous dimension of the field. This implies a change in the large-distance behavior of the correlation functions $\langle\phi(x)\phi(0)\rangle$ at T_c from the free-field behavior r^{-D+2} to $r^{-D+2-\eta}$. The results from the renormalization group are recovered from assumption (19). Comparison with Eq. (11) shows that $\eta_m = 2\gamma_m$, whereas η is the same as above.

Let us mention that this procedure leads to resummed expressions which have the same ε-expansions as those found by renormalization group techniques.

3.2 Five-Loop Results

In a first step, we determine the parameter ω such that the logarithmic derivative of $g(g_B)$ approaches zero for $g_B \to \infty$. We therefore insert the coefficients of the power series of $s(g_B)$ from Eq. (18) into Eq. (16) and determine $q = 2/\omega$ for $L = 2, 3, 4, 5$, such that $s_L^* = 0$. The resulting ε-expansion for the approach-to-scaling parameter ω reproduces the well-known ε-expansion [15] up to the corresponding order. In Fig. 1a), the approximations ω_L are plotted against the number of loops L for $\varepsilon = 1$ and $N = 3$. Apparently, the five-loop results are still some distance away from a constant $L \to \infty$-limit. The slow approach to the limit calls for a suitable extrapolation method. The convergence behavior in the limit $L \to \infty$ was determined [26] to be of the general form

$$f^*(L) \approx f^* + \text{const} \times e^{-c\,L^{1-\omega}}. \tag{21}$$

We plot the approximations s_L for a given ω near the expected critical exponent against L, and fit the points by the theoretical curve (21) to determine the limit s^*. Then ω is varied, and the plots are repeated until s^* is zero. The resulting ω is the desired critical exponent, and the associated plot is shown in Fig. 1b). Since the optimal variational parameter $\hat{g}_B$ comes from minima and

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

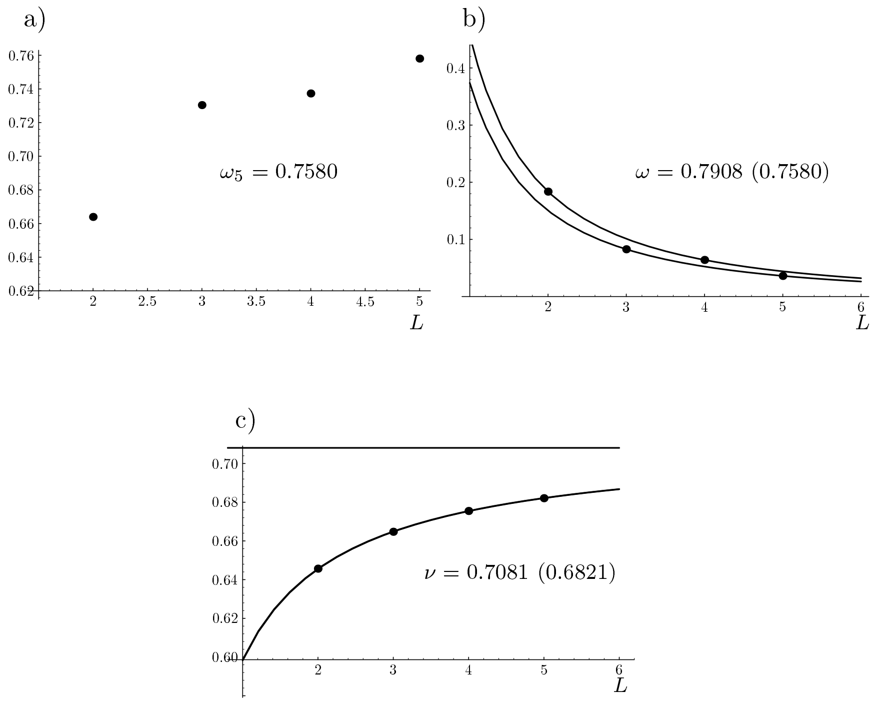

Figure 1. a) Critical exponent ω of approach to scaling calculated from $s_L^* = 0$, plotted against the order of approximation L for $N = 3$. b) Extrapolation of the solutions of the equation $s_L^* = 0$ to $L \to \infty$ with the help of Eq. (21). The value of ω for which $s_L^* \to 0$ for $L \to \infty$ determines $\omega = 2/q$. The best extrapolating function is $s_L = -6.8 \times 10^{-7} + 156.916e^{-5.84\,L^{0.2092}}$. c) Determination of the critical exponent ν plotted as a function of L. The extrapolating function is $\nu_L = 0.7081 - 4.0104e^{-3.6012\,L^{0.2092}}$, the horizontal line indicates the value of ν_∞.

turning points for even and odd approximants in alternate order, the points are best fitted by two different curves. The resulting ω-values are listed in Table 1. They are used to derive the strong-coupling limits for the exponents ν, γ and η. For the calculation of the critical exponent ν, we find the five-loop expansion for $\nu(g_B)$ using the relation $\nu(g_B) = 1/[2 - \eta_m(g_B)]$. From this we calculate the strong-coupling approximations ν_L for $L = 2, 3, 4, 5$. After extrapolating these to infinite L, we obtain the numbers listed for different universality classes $O(N)$ in Table 1. The corresponding extrapolation fits are

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Table 1. Critical exponents of five-loop strong-coupling theory and comparison with the results from Borel-type resummation (GZ) [33], and from variational perturbation theory in $D = 3$ dimensions [27]. The parentheses behind each number show the five-loop approximation to see the extrapolation distance.

	VPT, $D = 4 - \varepsilon$	Borel-Res. (GZ)	VPT $3D$
	$\omega_\infty(\omega_5)$		
$N = 0$	0.80345(0.7448)	0.828 ± 0.023	0.810
$N = 1$	0.7998(0.7485)	0.814 ± 0.018	0.805
$N = 2$	0.7948(0.7530)	0.802 ± 0.018	0.800
$N = 3$	0.7908(0.7580)	0.794 ± 0.018	0.797
	$\nu_\infty(\nu_5)$ (I)		
$N = 0$	0.5874(0.5809)	0.5875 ± 0.0018	0.5883
$N = 1$	0.6292(0.6171)	0.6293 ± 0.0026	0.6305
$N = 2$	0.6697(0.6509)	0.6685 ± 0.0040	0.6710
$N = 3$	0.7081(0.6821)	0.7050 ± 0.0055	0.7075
	$\eta_\infty(\eta_5)$ (I)		
$N = 0$	0.0316(0.0234)	0.0300 ± 0.0060	0.03215
$N = 1$	0.0373(0.0308)	0.0360 ± 0.0060	0.03370
$N = 2$	0.0396(0.0365)	0.0385 ± 0.0065	0.03480
$N = 3$	0.0367(0.0409)	0.0380 ± 0.0060	0.03447
	$\gamma_\infty(\gamma_5)$		
$N = 0$	1.1576(1.1503)	1.1575 ± 0.0050	1.616
$N = 1$	1.2349(1.2194)	1.2360 ± 0.0040	1.241
$N = 2$	1.31045(1.2846)	1.3120 ± 0.0085	1.318
$N = 3$	1.3830(1.3452)	1.3830 ± 0.0135	1.390

plotted in Fig. 1c). Similarly, estimations for the exponents η and γ can be obtained [19]. In Table 1 the resulting values are compared to those found by Borel resummation in $D = 4 - \varepsilon$ dimensions and by the same strong-coupling approach in $D = 3$ dimensions.

4 Conclusion

Instead of expressing the renormalization group functions in the renormalized coupling constant g_B, we can reexpand in the bare coupling constant g. This allows applying strong-coupling theory to the five-loop perturbation expan-

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

sions of $O(N)$-symmetric ϕ^4-theories in $4-\varepsilon$ dimensions. Satisfactory values for all critical exponents are obtained.

Acknowledgments

We thank the German Academic Exchange Service (DAAD) for support.

References

[1] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1980).
[2] K.G. Wilson, *Phys. Rev. B* **4**, 3184 (1971); K.G. Wilson and J.B. Kogut, *Phys. Rep. C* **12**, 75 (1974).
[3] K.G. Wilson and M.E. Fisher, *Phys. Rev. Lett.* **28**, 240 (1972).
[4] E. Brézin, J.C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, Vol. 6, Eds. C. Domb and M.S. Green (Academic Press, New York, 1976).
[5] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 1st ed. (Clarendon Press, Oxford, 1989).
[6] G. 't Hooft and M. Veltman, *Nucl. Phys. B* **44**, 189 (1972).
[7] G. 't Hooft, *Nucl. Phys. B* **61**, 455 (1973).
[8] E. Brézin, J.C. Le Guillou, J. Zinn-Justin, and B.G. Nickel, *Phys. Lett. A* **44**, 227 (1973).
[9] A.A. Vladimirov, D.I. Kazakov, and O.V. Tarasov, *Sov. Phys. JETP* **50**, 521 (1979); Preprint JINR E2-12249, Dubna, 1979.
[10] K.G. Chetyrkin, A.L. Kataev, and F.V. Tkachov, *Phys. Lett. B* **99**, 147 (1981); *ibid.* **101**, 457 (1981) (Erratum).
[11] D.I. Kazakov, *Phys. Lett. B* **133**, 406 (1983); *Teor. Mat. Fiz.* **58**, 343 (1984); *Dubna lecture notes*, E2-84-410 (1984).
[12] S.G. Gorishny, S.A. Larin, and F.V. Tkachov, *Phys. Lett. A* **101**, 120 (1984).
[13] K.G. Chetyrkin, S.G. Gorishny, S.A. Larin, and F.V. Tkachov, *Phys. Lett. B* **132**, 351 (1983); Preprint INR P-0453, Moscow, 1986.
[14] K.G. Chetyrkin and F.V. Tkachov, Preprint INR P-118, Moscow, 1979; K.G. Chetyrkin, A.L. Kataev, and F.V. Tkachov, *Nucl. Phys. B* **174**, 345 (1980).
[15] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S.A. Larin, *Phys. Lett. B* **272**, 39 (1991); H. Kleinert and V. Schulte-

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

Frohlinde, *Phys. Lett. B* **342**, 284 (1995).
[16] H. Kleinert, S. Thoms, and V. Schulte-Frohlinde, *Phys. Rev. B* **56**, 14428 (1997).
[17] H. Kleinert, *Phys. Lett. B* **434**, 74 (1998).
[18] H. Kleinert, *Phys. Lett. B* **463**, 69 (1999).
[19] H. Kleinert and V. Schulte-Frohlinde, *J. Phys. A* **34**, 1037 (2001), eprint: cond-mat/9907214.
[20] H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of ϕ^4-Theories* (World Scientific, Singapore, 2001).
[21] A.A. Vladimirov, *Teor. Mat. Fiz.* **43**, 210 (1980).
[22] F.V. Tkachov, *Phys. Lett. B* **100**, 65 (1981); K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys. B* **192**, 159 (1981).
[23] K.G. Chetyrkin and F.V. Tkachov, *Phys. Lett. B* **114**, 340 (1982); K.G. Chetyrkin and V.A. Smirnov, *Phys. Lett. B* **144**, 410 (1984).
[24] D.J. Broadhurst, *Massless Scalar Feynman Diagrams: Five Loops and Beyond*, Open University Preprint OUT - 4102 - 18 1985, Milton Keynes, U.K.
[25] D.I. Kazakov and D.V. Shirkov, *Fortschr. Phys.* **28**, 465 (1980); J. Zinn-Justin, *Phys. Rep.* **70**, 3 (1981).
[26] H. Kleinert, *Phys. Rev. D* **57**, 2264 (1998); Add.: *ibid.* **58**, 1077 (1998).
[27] H. Kleinert, *Phys. Rev. D* **60**, 85001 (1999).
[28] H. Kleinert, *Phys. Lett. A* **173**, 332 (1993).
[29] Details of strong-coupling theory are found in Chapter 5 of the textbook: H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995).
[30] W. Janke and H. Kleinert, *Phys. Lett. A* **199**, 287 (1995).
[31] W. Janke and H. Kleinert, *Phys. Rev. Lett.* **75**, 2787 (1995).
[32] H. Kleinert and W. Janke, *Phys. Lett. A* **206**, 283 (1995).
[33] R. Guida and J. Zinn-Justin, *J. Phys. A* **31**, 8103 (1998).
[34] H. Kleinert, *Phys. Lett. A* **207**, 133 (1995).