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ASYMPTOTIC FREEDOM IN CURVATURE-SATURATED GRAVITY

S. CAPOZZIELLO, G. LAMBIASE

Dipartimento di Scienze Fisiche "E.R. Caianiello",
Università di Salerno, 84081 Baronissi (Sa), Italy and
Istituto Nazionale di Fisica Nucleare, Sez. di Napoli, Italy
E-mails: capozziello@sa.infn.it, lambiase@sa.infn.it

H.-J. SCHMIDT

Freie Universität Berlin, Institut für Theoretische Physik,
Arnimallee 14, D-14195 Berlin, Germany and
Institut für Mathematik, Universität Potsdam,
Am Neuen Palais 10, D-14469 Potsdam, Germany
E-mail: hjschmi@rz.uni-potsdam.de

For a spatially flat Friedmann model with line element $ds^2 = a^2[da^2/B(a) - dx^2 - dy^2 - dz^2]$, the 00-component of the Einstein field equation reads $8\pi G T_{00} = 3/a^2$ and contains no derivative. For a nonlinear Lagrangian $\mathcal{L}(R)$, we obtain a second–order differential equation for B instead of the expected fourth-order equation. We discuss this equation for the curvature-saturated model proposed by Kleinert and Schmidt [1]. Finally, we argue that asymptotic freedom $G_{\text{eff}}^{-1} \to 0$ is fulfilled in curvature-saturated gravity.

1 Introduction

In the past decades, several extended theories of gravity have been proposed, whose effective actions are more general than the Einstein-Hilbert action. This approach is motivated by unification schemes which consider gravity at the same level as the other interactions of elementary particles. In such theories, we define an effective gravitational coupling G_{eff} and an effective cosmological constant Λ_{eff} which give the observed values $G_{\text{eff}} \to G$ and $\Lambda_{\text{eff}} \to \Lambda$ in the weak-energy limit.

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These extended theories introduce new features into gravitational physics which general relativity does not possess, in particular higher-order terms in curvature invariants or non-minimal couplings between geometry and scalar fields. For example, *asymptotic freedom* can be related to singularity-free cosmological models. Its emergence could be the result of the fact that gravity is "induced" by an average effect of the other interactions [2,3].

However, the concept of gravitational asymptotic freedom is not completely analogous to that in non-abelian gauge theories of strong interaction, since a full quantum theory of gravity does not yet exist. So far, asymptotic freedom has been shown for several classes of gravitational Lagrangians which are not of physical interest.

Another interesting feature is that, by extended theories of gravity, cosmological singularities can be avoided introducing a *limiting curvature hypothesis* for values of curvature near the Planck scale [4]. This fact is particularly relevant for string-dilaton gravity since, by duality, it allows to recover large classes of cosmological solutions for $t \to -\infty$ [5].

In Ref. [1], it was argued that the Lagrangian for gravity should remain bounded at large curvature. One example for such a behavior is the curvature-saturated Lagrangian

$$\mathcal{L}_{\rm CS} = \frac{1}{16\pi G} \frac{R}{\sqrt{1 + l^4 R^2}} \,. \tag{1}$$

It has been discussed in Ref. [1] with methods developed for the study of fourth-order gravity developed in Refs. [6,7] and the publications cited therein. In Eq. (1), l is a length parameter, and for $l = 0$, $\mathcal{L}_{\rm CS}$ reduces to the Einstein-Hilbert Lagrangian. Earlier models with a nonlinear Lagrangian $\mathcal{L}(R)$ made only a polynomial approximation like

$$\mathcal{L} = \frac{R}{16\pi G} + \sum_{k=2}^{n} c_k R^k \,, \quad c_n \neq 0, \tag{2}$$

sometimes accompanied by a logarithmic or a R^m-term with non-integer m. In all such cases, one had $d\mathcal{L}/dR \to \pm\infty$ for $|R| \to \infty$. In contrast to this, $\mathcal{L}_{\rm CS}$ of Eq. (1) has the behavior $d\mathcal{L}/dR \to 0$ for $R \to \pm\infty$.

This limiting property can be reformulated in terms of the effective gravitational coupling

$$G_{\rm eff} = \left[16\pi \frac{d\mathcal{L}}{dR} \right]^{-1} \tag{3}$$

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as follows: In the previously extended models (2), G_{eff} is bounded (including zero), in contrast to the curvature-saturated model (1) where one gets $|G_{\text{eff}}| \to \infty$ as $|R| \to \infty$; of course, for $l = 0$ we recover $G_{\text{eff}} \equiv G$.

Other limiting behaviors have been discussed in Refs. [2,3]. In Ref. [3], a scalar-tensor theory with Lagrangian

$$\mathcal{L}_\varphi = F(\varphi)R + \frac{1}{2}g^{ij}\varphi_{,i}\varphi_{,j} - V(\varphi) \tag{4}$$

has been used where $i, j = 0, 1, 2, 3$ and the effective gravitational constant is

$$G_{\text{eff}} = \frac{1}{16\pi F(\varphi)} \tag{5}$$

instead of our Eq. (3). In Ref. [2], G_{eff} was assumed to depend on the matter density in such a way that it vanishes for high density, thus leading to asymptotic freedom at high energies.

In Ref. [8], possible finite-size *Casimir* effects to the free energy have been calculated for massive and massless scalar fields, which can produce a quantum-effected effective gravitational constant.

From a geometric point of view, one may argue as follows: For large curvatures, the gravitational action $I = \int \mathcal{L}_{\text{C}}\sqrt{-g}d^4x$ shall not depend on any scale; the simplest Lagrangian leading to such a behavior is

$$\mathcal{L}_{\text{C}} = C_{ijkl}C^{ijkl}\,, \tag{6}$$

the Lagrangian of the conformally invariant Weyl gravity (see Ref. [9] and the references therein).

Here, we extend the discussion of Ref. [1] to more general types of curvature-saturated Lagrangians than Eq. (1), and we give more details about the set of spatially flat Friedmann models solving the corresponding field equations. As it will become clear below, the concept of asymptotic freedom is meaningful also for curvature-saturated theories and it is necessary to define more specifically the notion of an effective cosmological constant.

2 Deduction of the Field Equations

For the action $I = \int \mathcal{L}(R)\sqrt{-g}d^4x$, it is useful to write the metric of the expanding spatially flat Friedmann model as

$$ds^2 = a^2\left[\frac{da^2}{B(a)} - dx^2 - dy^2 - dz^2\right]\,, \tag{7}$$

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with $a > 0$ and $B(a) > 0$ as shown in Ref. [1]. In these coordinates, the parameter a is called *curvature time* because the curvature scalar has the simple form

$$R = -\frac{3}{a^3}\frac{dB}{da}\,, \tag{8}$$

which is linear in the only unknown function $B(a)$, and it does not contain second derivatives. The non-vanishing components of the Christoffel affinity are as follows

$$\Gamma^0_{00} = \frac{1}{a} - \frac{B'}{2B}\,, \quad \Gamma^\beta_{0\alpha} = \frac{1}{a}\delta^\beta_\alpha\,, \quad \Gamma^0_{\alpha\beta} = \frac{B}{a}\delta_{\alpha\beta}\,, \tag{9}$$

where the dash denotes d/da. The greek indices assume the spatial values $\alpha, \beta = 1, 2, 3$, and $\sqrt{-g} = a^4/\sqrt{B}$. The Ricci tensor has the components

$$R_{\alpha\beta} = \left(\frac{B}{a^2} + \frac{B'}{2a}\right)\delta_{\alpha\beta}\,, \quad R_{00} = \frac{3}{a^2} - \frac{3B'}{2aB}. \tag{10}$$

Together with the metric (7), its mixed-variant version can be calculated to

$$R^\beta_\alpha = -\left(\frac{B}{a^4} + \frac{B'}{2a^3}\right)\delta^\beta_\alpha\,, \quad R^0_0 = \frac{3B}{a^4} - \frac{3B'}{2a^3}\,. \tag{11}$$

The most often used direct way to deduce the field equation is to insert these expressions into the fourth-order field equation $\delta I/\delta g_{ij} = 0$ following from the variation of the action I.

Here we present a shorter and more direct derivation: We first use the fact that the field equation also implies $\delta I/\delta B = 0$, and in a second step, verify that in spite of this simplified variation, no spurious solutions appear.

Let us denote $d\mathcal{L}/dR$ by $h(R)$, and $d^2\mathcal{L}/dR^2$ by $k(R)$. The vacuum field equation reads

$$0 = \frac{\delta(\mathcal{L}\sqrt{-g})}{\delta B} = \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial B} - \left(\frac{\partial(\mathcal{L}\sqrt{-g})}{\partial B'}\right)'\,.$$

After multiplying this by a^7 to avoid negative a-powers, we obtain

$$\begin{aligned} 0 = &-a^7\mathcal{L}(-3B'/a^3) + 3a^3(2B - aB')h(-3B'/a^3) + \\ &+18B(3B' - aB'')k(-3B'/a^3)\,. \end{aligned} \tag{12}$$

It is remarkable that this equation is of second order for one function B only, but nevertheless, it is equivalent to the whole fourth-order field equation for

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the metric (7). One order reduction follows from Eq. (8), the other from the fact that Eq. (12) is a constraint and not the full dynamical equation.

To exclude the existence of spurious solutions it now suffices to see that exactly two free initial conditions can be put: $B(a_0)$ and $B'(a_0)$.
Example: Let $\mathcal{L} = R + 2\Lambda$, where Λ is a constant. Then $h = 1$ and $k = 0$, and Eq. (12) reads

$$B(a) = \frac{a^4 \Lambda}{3} \,. \tag{13}$$

The Hubble parameter H is related to B via

$$B(a) = H^2 a^4 \tag{14}$$

(see Ref. [1]). So, with Eq. (13) we get $\Lambda = 3H^2 = \text{const.}$, consistent with the usual de Sitter space-time calculation.

3 Solutions of the Field Equations

Now we consider some more general cases for $\mathcal{L}(R)$ and look for the corresponding solutions of Eq. (12).

3.1 Solution for Einstein's Theory

To get a feeling for the coordinates (7) we look for the 00-component of the Einstein field equation

$$8\pi G T_{ij} = R_{ij} - \frac{1}{2} R g_{ij} \,.$$

Using Eqs. (8) and (10), we obtain

$$8\pi G T_{00} = \frac{3}{a^2} \,. \tag{15}$$

The energy density is always non-negative for spatially flat Friedmann models in Einstein's theory. Eq. (15) provides us with another justification for calling this a curvature-coordinate: a is chosen such that T_{00} does not depend at all on the function $B(a)$.

Raising one index of Eq. (15) we find

$$8\pi G \rho \equiv 8\pi G T_0^0 = \frac{3B}{a^4} \,. \tag{16}$$

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Together with Eq. (14), this represents a good consistency test: it yields the Friedmann equation

$$8\pi G\rho = 3H^2\,, \tag{17}$$

i.e. curvature time and synchronous time give identical results.

For the equation of state $p = \alpha\rho$ we have

$$\rho a^{3(1+\alpha)} = \text{const.} \tag{18}$$

Together with Eq. (16), this yields

$$B = B_0 a^{1-3\alpha}\,, \tag{19}$$

with a positive constant B_0. This is consistent with Eq. (13) for $\alpha = -1$.

3.2 *Solution for $\mathcal{L} = R^m$*

For the Lagrangian $\mathcal{L} = R^m$, with a constant m ($m \neq 0, 1$), Eq. (12) simplifies to

$$0 = 2m(3m-4)BB' + (m-1)aB'^{\,2} - 2m(m-1)aBB''\,. \tag{20}$$

We insert

$$B = \exp\left(\int \beta da\right)\,. \tag{21}$$

The integration constant in Eq. (21) need not be specified, because Eq. (20) implies that with $B(a)$ also $CB(a)$, with any positive constant C, represents a solution.

Eq. (21) implies $B' = \beta B$ and $B'' = (\beta' + \beta^2)B$. Then Eq. (20) becomes

$$0 = 2m(3m-4)\beta - (m-1)(2m-1)a\beta^2 - 2m(m-1)a\beta'\,. \tag{22}$$

Here we insert $\beta = \gamma/a$, $\beta' = \gamma'/a - \gamma/a^2$ and get

$$0 = 2m(4m-5)\gamma - (m-1)(2m-1)a\gamma^2 - 2m(m-1)a\gamma'\,. \tag{23}$$

By putting $x = \ln a$, Eq. (23) can be rewritten in the form

$$\frac{d\gamma}{dx} = \gamma\left[\frac{4m-5}{m-1} - \frac{2m-1}{2m}\gamma\right]\,. \tag{24}$$

The de Sitter space-time is represented by $\gamma \equiv 4$. From Eq. (24) it becomes clear that it is a solution for $m = 2$ only, i.e. for $\mathcal{L} = R^2$.

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All solutions of (24) can be given in closed form. For $m = 5/4$, we get with any constant x_0:

$$\gamma = \frac{5}{3(x - x_0)} \ ,$$

and, together with $a_0 = e^{x_0}$ and Eq. (21), finally

$$B(a) = \exp\left[\frac{5}{3}\int \frac{da}{a\ln(a/a_0)}\right] = B_0\left(\ln\frac{a}{a_0}\right)^{5/3} . \tag{25}$$

For $m = 1/2$ we find $\gamma = \pm e^{6(x-x_0)}$ and

$$B(a) = \exp[\pm(a/a_0)^6] \, . \tag{26}$$

Let us study the stability of these solutions. Instead of directly comparing with neighboring functions $B(a)$ it is easier to consider the neighborhood of the solution: $\gamma =$ const. in Eq. (24). Up to the uninteresting solution $\gamma = 0$ representing $R \equiv 0$ we find

$$\gamma = \gamma_0 \equiv \frac{4m-5}{m-1}\frac{2m}{2m-1},$$

i.e.

$$\frac{d\gamma}{dx} = \gamma(\gamma_0 - \gamma)\frac{2m-1}{2m} \, . \tag{27}$$

For $m > 1/2$ this implies stability. For $m = 2$ this proves again the attractor property of the de Sitter space-time.

In synchronized time, power-law inflation is described by $a(t) \sim t^n$, $n \geq 1$. In our coordinates, such solutions correspond to a constant γ within the interval $2 \leq \gamma < 4$.

3.3 Solutions for Curvature-Saturated Lagrangians

As another model, consider the high-curvature ansatz

$$\mathcal{L} = \Lambda + \frac{C}{R} \, , \tag{28}$$

in which $h(R) = -C/R^2$, $k(R) = 2C/R^3$, with constants Λ and C. The ansatz (28) is supposed to approximate the regions $R \to \pm\infty$, and it will be matched at small curvature values to a polynomial Lagrangian of type Eq. (2). Here we are only interested in the high-curvature regions. The concrete values

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of Λ and C are not yet fixed, and they may be different for $R \to +\infty$ and $R \to -\infty$, respectively.

For $\Lambda = 0$ we can directly apply Eq. (27) with $m = -1$, i.e. $\gamma_0 = 3$, and

$$\frac{d\gamma}{dx} = \frac{3}{2}\gamma(3-\gamma)\,. \tag{29}$$

This equation can be integrated in closed form, but we only need that $\gamma = 3$ represents power-law inflation and that this represents an attractor solution. In synchronized coordinates, it reads

$$ds^2 = dt^2 - t^4(dx^2 + dy^2 + dz^2)\,, \tag{30}$$

giving $R \to 0$ as $t \to \infty$. Thus, for sufficiently large t, the development leads to our Universe today.

For $\Lambda \neq 0$, we insert (28) into (12), and have to solve

$$3\Lambda(B')^3 = 2a^2C[5BB' + aB'^{\,2} - 2aBB'']\,. \tag{31}$$

This equation will be further integrated in future work.

4 Curvature-Saturated Gravity with Matter

To include matter, we have to replace the l.h.s. of Eq. (12) by $a^7\rho$. For an ideal fluid, we can use Eq. (18) as an equation of state. As a test of this procedure, we can insert $\mathcal{L} = R/16\pi G$, and get $a^7\rho = 3a^3B/8\pi G$, which is consistent with Eq. (16).

Therefore, matter with the equation of state $p = \alpha\rho$ can be described by $\rho = \rho_0 a^{-3(1+\alpha)}$, and the equations of motion are

$$\begin{aligned} a^7\rho &= -a^7\mathcal{L}(R) + 3a^3(2B - aB')h(R) + 18B(3B' - aB'')k(R)\,, \\ R &= -3B'a^{-3}\,. \end{aligned} \tag{32}$$

We stress that such equations hold for general $\mathcal{L}(R)$-Lagrangians and for any perfect fluid matter.

5 Conclusions

In this paper, we continued the discussion on properties of the curvature-saturated cosmological models proposed in Ref. [1]. Especially, we deduced the main equation *with* matter, Eq. (32), for a spatially flat Friedmann model in the curvature coordinates (7).

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We have shown that, from the singularity, the Universe expands via power-law inflation (which represents a transient attractor) to the actual state.

In order to find the Wheeler-DeWitt equation for fourth-order gravity models (see Refs. [6,10,11] and the papers cited there), one usually has to introduce, in a more or less natural way, more degrees of freedom to reduce the equation down to second order. Here we presented a version where the Lagrangian can be directly used: $\mathcal{L}(R)\sqrt{-g}$ depends on a, B and B' only; of course, different from the use of the synchronized time coordinate, we have now an explicit dependence of the metric coefficients on the time-like coordinate a.

As a final remark, we see that asymptotic freedom $G_{\text{eff}}^{-1} = 16\pi d\mathcal{L}/dR \to 0$ can easily be incorporated also in curvature-saturated gravity but its meaning is different from that in Ref. [3]. There G_{eff} is a function of matter density which regulates the gravitational coupling, here it is the scalar curvature (i.e. the form of the gravitational Lagrangian) which leads towards the saturation and then towards asymptotic freedom. Future studies will be devoted to more physically motivated effective Lagrangians.

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