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NAMBU-GOTO STRING WITHOUT TACHYONS BETWEEN A HEAVY AND A LIGHT QUARK

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We point out that the singularity in the interquark potential at small distances disappears in infinite space-time dimensions if a Nambu-Goto string is anchored at one end to an infinitely heavy quark and at the other to an infinitely light quark. This suggests that, if such quarks are placed at the ends of the string, some unphysical features such as tachyon states in the string spectrum are absent also in finite dimensions.

1 Introduction

We consider it a great honour presenting a contribution to the *Festschrift* dedicated to Professor Hagen Kleinert on the occasion of his 60th birthday. We were lucky having a very fruitful and instructive collaboration with him in our studies of string dynamics, when we had the opportunity to profit from Kleinert's intuition and his nonstandard approaches to complicated problems of theoretical physics.

Here we present the results of such a collaboration when we studied the effect of the boundary conditions imposed on dynamical variables (string coordinates) on the Nambu-Goto string spectrum. It is difficult to overestimate

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the role of Professor Kleinert in obtaining these results which are in main part due to his skill of deducing far going physical inferences from the mathematical facts derived.

It is generally believed that some modification of the Nambu-Goto string model will eventually become a fundamental theory capable of explaining the forces between quarks in a simpler way than quantum chromodynamics (QCD). Indeed, the correct large-distance confinement behavior [1–3] is automatically obtained, by construction, whereas that in QCD can only be found by arduous lattice simulations [4,5]. Also the first quantum correction to this behavior, the universal Lüscher term [1,6], is found immediately. It is a one-loop contribution to the string energy and corresponds to the zero point energy of the small oscillations, coinciding with the Casimir energy at $T = 0$.

Certainly, it cannot be hoped that the Nambu-Goto string is anywhere close to the real color-electric flux tube between quarks since it is uncapable of reproducing the $1/R$-singularity at small R caused by the asymptotic freedom of gluons. Some essential modification accounting for the finite diameter of the flux tube will be necessary, in particular its transition into a spherical bag at small quark separations. A first attempt in this direction was taken some time ago by adding an asymptotically-free curvature stiffness term [7], but this term introduced other problems. In particular, the true stiffness constant of the flux tube appears with the opposite sign [8].

In spite of the essential differences between a Nambu-Goto string and a flux tube connecting quarks, the question arises how the unphysical properties of a Nambu-Goto string change if quarks are placed at its ends. The purpose of this note is to point out that in one extremal configuration, at least the singularity of the string potential disappears, indicating the absence of tachyons in that case [9].

To obtain a first idea about all properties of a fully fluctuating string, it is useful to investigate the limit of infinite space-time dimension D, where a saddle-point approximation to the functional integral yields exact results via one-loop calculations. This limit exhibits immediately an important unphysical feature of the Nambu-Goto string model [1]. There a complex string potential appears at distances smaller than a critical radius R_c, where the interquark potential vanishes. The existence of such a critical radius is attributed to tachyonic states in the string spectrum [10].

This and other unphysical properties are found in the so-called *static* interquark potential, where the string is anchored to immobile infinitely heavy

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quarks. In this limit, the eigenfrequencies of the string are $\omega_n = n\pi/R$, $n = 1, 2, \ldots$, where R is the distance between the quarks. The associated Casimir energy reads

$$E_{\rm C} = \frac{D-2}{2}\sum_{n=1}^{\infty}\omega_n = -\frac{\pi(D-2)}{24\,R}, \tag{1}$$

yielding the well-known Lüscher term. Here Riemann's zeta function has been used [11].

We shall see in Section 3 that the Casimir energy determines the interquark potential completely in the limit $D \to \infty$, yielding

$$V(R) = M_0^2\,R\sqrt{1+\frac{\bar{R}^2}{R^2}}, \qquad \bar{R}^2 = \frac{2RE_{\rm C}}{M_0^2}. \tag{2}$$

Inserting (1), the potential calculated by Alvarez follows [1]

$$V_{\rm Alvarez} = M_0^2 R\sqrt{1-\frac{R_c^2}{R^2}}, \quad R_c^2 = \frac{\pi(D-2)}{12\,M_0^2}. \tag{3}$$

The quantity M_0^2 is the string tension. The same potential is found for strings with free ends due to the same Casimir energy (1).

2 Interquark Potential Generated by a String with Massive Ends

If a Nambu-Goto string has point-like quarks of masses m_1, m_2 at the ends moving along the worldlines $C_a, a = 1, 2$, the action reads [12] ($\hbar = c = 1$):

$$\mathcal{A} = -M_0^2\iint d^2\xi\sqrt{g} - \sum_{a=1}^{2} m_a\int_{C_a} ds_a\,, \tag{4}$$

where $g = \det(g_{\alpha\beta})$ is the determinant of the string metric. If the string coordinates are parameterized by $x^\mu(\xi)$, then $g_{\alpha\beta} = \partial_\alpha x^\mu\,\partial_\beta x_\mu\,,\ \alpha,\beta = 0,1$. For calculating the interquark potential from such an action one conveniently uses the Gauss parameterization:

$$x^\mu(t,r) = (t, r, \mathbf{u}(t,r)), \quad 0 \le r \le R, \quad \mu = 0, 1, \ldots, D-1\,, \tag{5}$$

where the vector field $\mathbf{u}(t,r) = (x^2(t,r), \ldots, x^{D-1}(t,r))$ describes the transverse displacements of the string in D dimensions. Then $g_{\alpha\beta} = \delta_{\alpha\beta} +$

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$\partial_\alpha \mathbf{u} \partial_\beta \mathbf{u}$, with $\mathbf{uu} \equiv \sum_{j=2}^{D-1} u^j\, u^j$. The fluctuation spectrum is found from the linearized equations of motion and boundary conditions

$$\Box \mathbf{u} \,=\, 0\,, \tag{6}$$

$$m_1\, \ddot{\mathbf{u}} \,=\, M_0^2\, \mathbf{u}', \quad r \,=\, 0\,, \tag{7}$$

$$m_2\, \ddot{\mathbf{u}} \,=\, -\, M_0^2\, \mathbf{u}', \quad r \,=\, R\,. \tag{8}$$

Here dots and primes denote the derivatives with respect to t and r, respectively, and $\Box \equiv \partial^2/\partial t^2 \,-\, \partial^2/\partial r^2$. The general solution to these equations has the form

$$u^j(t,r) \,=\, i\, \frac{\sqrt{2}}{M_0^2} \sum_{n\neq 0} e^{-i\omega_n t}\, \frac{\alpha_n^j}{\omega_n}\, u_n(r), \quad j \,=\, 2,\ldots,D-1\,, \tag{9}$$

where the amplitudes α_n^j satisfy the usual rule of the complex conjugation, $\alpha_n \,=\, \alpha^*_{-n}$. The unnormalized eigenfunctions $u_n(r)$ are

$$u_n(r) \,=\, \cos\omega_n r \,-\, \omega_n \frac{m_1}{M_0^2} \sin\omega_n r\,, \tag{10}$$

and the eigenfrequencies ω_n satisfy the secular equation

$$\tan\omega R \,=\, \frac{M_0^2\,(m_1 \,+\, m_2)\omega}{m_1 m_2 \omega^2 \,-\, M_0^4}\,. \tag{11}$$

The Hamiltonian operator reads

$$H \,=\, \sum_n \sum_{j=2}^{D-1} \omega_n\, a_n^{j\dagger} a_n^j \,+\, E_{\rm C}, \tag{12}$$

where $E_{\rm C}$ is the Casimir energy

$$E_{\rm C} = \frac{D-2}{2} \sum_n \omega_n\,. \tag{13}$$

The creation and annihilation operators satisfy the usual commutation rules

$$[a_n^i, a_m^{j\dagger}] \,=\, \delta^{ij}\, \delta_{nm}\,. \tag{14}$$

The Casimir energy [13,14] renders the Lüscher correction to the interquark potential [6].

As in all field theories [14], the Casimir energy $E_{\rm C}$ of a string diverges for large n, and a renormalization is necessary to obtain physical results. If both masses are infinite or zero, the roots in Eq. (11) are $n\pi/R$ with integer n, and

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the sum over eigenvalues is made finite with the help of the zeta function [11] in Eq. (1).

The interesting alternative situation is the limiting case, $m_1 = \infty$ and $m_2 = 0$, in which one end is fixed, the other free. Such a string approximates mesons consisting of one heavy and one light quark bound together by a color-electric flux tube. In this limit, the boundary conditions (7) and (8) simplify to

$$\mathbf{u}(t,0) = 0, \quad \mathbf{u}'(t,R) = 0\,, \tag{15}$$

and the secular equation (11) assumes the form

$$\cos\omega R = 0\,, \tag{16}$$

which is solved by string eigenfrequencies ω_n being half-integer multiples of π/R: $\omega_n = (n+1/2)\pi/R$ for $n = 0, 1, \ldots$. Then, the Casimir energy is

$$E_{\mathrm{C}} = \frac{\pi(D-2)}{48\,R}\,, \tag{17}$$

where again the Riemann zeta function has been used [11,15]. The Casimir energy has now a *positive* sign, and half the magnitude, and Eq. (2) yields the interquark potential

$$V^{(R)} = M_0^2 R\sqrt{1 + \frac{1}{2}\frac{R_c^2}{R^2}}, \quad R_c^2 = \frac{\pi(D-2)}{12\,M_0^2}. \tag{18}$$

This is an important result because the interquark potential (18) is real for all distances R in the limit $D \to \infty$. This implies that a string with these boundary conditions is a physical model for all distances R (certainly, again in the limit $D \to \infty$). Figure 1 compares the new string potential which is physical for all distances R with the Alvarez potential which is real only for $R > R_c$. This observation raises the question whether there might be an entire regime of asymmetric quark mass configurations for which the potential remains physical. We intend to study the general case of both masses being finite. Then the roots in Eq. (11) have the large-n behavior

$$\omega_n \simeq n\,\frac{\pi}{R} + \frac{M_0^2(m_1 + m_2)}{m_1\,m_2}\frac{1}{n\pi} + O(n^{-3})\,. \tag{19}$$

The formal zeta function regularization can no longer be applied (since $\sum_{n=1}^{\infty} n^{-1} = \zeta(1) = \infty$), calling for a different and more physical subtraction procedure.

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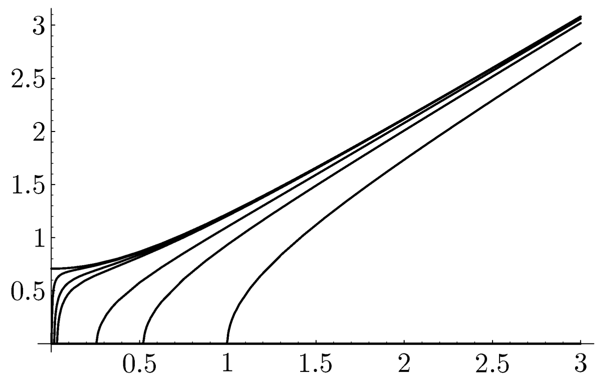

Figure 1. Dependence of the dimensionless interquark potential on the boundary conditions in the string model. The upper curve shows the potential (18) corresponding to the extremely asymmetric boundary condition of one string end being fixed, and the other free ($m_1 = \infty$, $m_2 = 0$). The lowest curve presents Alvarez' result (3) for strings with both ends fixed or free ($m_1 = \infty$, $m_2 = \infty$). The lengths are measured in units of Alvarez' critical radius R_c. The remaining curves show the potential for one infinite and one finite quark mass m_2 corresponding to the reduced mass parameter $\rho_2 = 0,\ 1/5,\ 1,\ 2,\ 10,\ 100,\ \infty$ (from lowest to highest curves).

There exists a simple analytic expression for the subtracted Casimir energy [16,17]. To find it we introduce the dimensionless frequency sum $S \equiv (12R/\pi)\sum_n \omega_n$ and rewrite it as

$$S = \frac{6Ri}{\pi^2}\int d\omega\,\omega\frac{d}{d\omega}\log[\cos(\omega R)M_0^2(m_1+m_2)\omega - \sin(\omega R)(m_1m_2\omega^2 - M_0^4)] - (R\to\infty). \tag{20}$$

The derivative of the logarithm contains the solutions of the secular equation (11) as poles with unit residue. The contour of integration encloses the positive ω-axis in the clockwise sense. After opening up the contour and integrating along the imaginary frequency axis $\omega = iy$, a partial integration leads to

$$S = \frac{6R}{\pi^2}\int_{-\infty}^{\infty} dy\log[\cosh(yR)M_0^2(m_1+m_2)y + + \sinh(yR)(m_1m_2y^2 + M_0^4)] - (R\to\infty). \tag{21}$$

For a comparison of the behavior of the quark potential for various quark mass

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configurations it is useful to go over to the dimensionless distance variable $\rho \equiv R/R_c$ and to reduced quantities $\rho_{1,2} \equiv R_{1,2}/R_c$, where $R_{1,2}$ are length parameters associated with the quark masses defined by

$$R_{1,2} \equiv \frac{\pi(D-2)}{12m_{1,2}}. \tag{22}$$

With the integration variable $z = yR$, we can rewrite S as

$$\begin{aligned} S(\rho) &= \frac{12}{\pi^2}\int_0^\infty dz \log\left[1 - e^{-2z}h(z,\rho)\right], \\ h(z,\rho) &= \frac{z^2-(\rho_1+\rho_2)\rho\, z + \rho_1\rho_2\rho^2}{z^2+(\rho_1+\rho_2)\rho\, z + \rho_1\rho_2\rho^2}. \end{aligned} \tag{23}$$

For $m_1 = \infty$, i.e. $\rho_1 = 0$, $S(\rho)$ is a simple function of $\rho_2\rho$ which runs from $S = -1$ for $\rho_2\rho = 0$ to $S = 1/2$ for $\rho_2\rho = \infty$. In terms of $S(\rho)$, the interquark potential acquires the general form [16,18,19]

$$\frac{V}{M_0^2 R_c} = \rho\sqrt{1 + \frac{S(\rho)}{\rho^2}}. \tag{24}$$

In Fig. 1 we have plotted the potential for $\rho_1 = 0$ and different $\rho_2 = 0,\ 1/5,\ 1,\ 2,\ 10,\ 100,\ \infty$. The plot shows that only the limit $m_2 = 0$ is associated with a real interquark potential for all R. For a small but finite m_2, the function $S(\rho)$ always becomes negative if the radius R is much smaller than m_2/M_0^2.

3 Functional Integral for String Potential

Let us verify that the interquark potential is indeed determined by the Casimir energy as stated in Eq. (2). The potential $V(R)$ between massive quarks separated by a distance R is defined by the functional integral [6,20,21]

$$e^{-TV(R)} = \int [D\mathbf{u}]\, e^{-\mathcal{A}_E[\mathbf{u}]}, \quad T \to \infty, \tag{25}$$

where $\mathcal{A}_E$ is the Euclidean action

$$\mathcal{A}_E = M_0^2 \int_0^T dt \int_0^R dr \sqrt{\det(\delta_{\alpha\beta} + \partial_\alpha \mathbf{u}\, \partial_\beta \mathbf{u})} +$$

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$$+\sum_{a=1}^{2} m_a \int_0^T dt\, \sqrt{1+\dot{\mathbf{u}}^2(t,r_a)}\,. \tag{26}$$

We want to calculate the leading term for $D \to \infty$. As usual, we make the action harmonic in the string positions by introducing auxiliary composite fields $\sigma_{\alpha\beta}$ and by constraining these to be equal to $\partial_\alpha \mathbf{u}\, \partial_\beta \mathbf{u}$ by means of a Lagrange multiplier $\alpha^{\alpha\beta}$. By a similar manipulation, also the end-point actions can be made harmonic. After some manipulations, the functional integral (25) becomes Gaussian in $\mathbf{u}$ and can be performed with the result

$$e^{-TV(R)} = \int [D\alpha][D\sigma]\, e^{-\mathcal{A}_E[\alpha,\sigma]}, \quad T \to \infty\,, \tag{27}$$

where

$$\mathcal{A}_E = M_0^2 \int_0^T dt \int_0^R dr \left[\sqrt{\det(\delta_{\alpha\beta}+\sigma_{\alpha\beta})} - \frac{1}{2}\alpha^{\alpha\beta}\sigma_{\alpha\beta}\right] + \\ + \frac{D-2}{2}\, \mathrm{Tr}\ln(-\partial_\alpha \alpha^{\alpha\beta}\partial_\beta)\,. \tag{28}$$

The boundary term in (26) is taken into account via the eigenvalues of the differential operator $-\partial_\alpha \alpha^{\alpha\beta}\partial_\beta$ in the action (28). As in Ref. [1], the functional integral is determined by the stationary point of (28) at which the matrices α and σ are diagonal. This simplifies the functional trace in (28) which becomes

$$\frac{D-2}{2}\mathrm{Tr}\ \ln(-\partial_\alpha \alpha^{\alpha\beta}\partial_\beta) = T\sqrt{\frac{\alpha^{11}}{\alpha^{00}}}\, E_{\mathrm{C}}. \tag{29}$$

Extremizing (28) with respect to $\sigma_{00}, \sigma_{11}, \alpha^{00}, \alpha^{11}$ yields indeed the string potential (2), as stated above.

4 Closing Remarks

It will be interesting to see whether the results derived in this note are present also in a finite dimension D. If this is so, then at least the limiting asymmetric quark mass configuration may be free of some of the unphysical features of present-day string models.

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Finally we note that a dependence of the interquark potential on the quark masses at the ends was observed before in different ways [4,5]. In quantum field theory, the influence of different boundary conditions upon the Casimir effect has also been explored [14] resulting in energies of opposite signs.

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