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FAST-CONVERGENT RESUMMATION ALGORITHM AND CRITICAL EXPONENTS OF ϕ^4-THEORY IN THREE DIMENSIONS

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During 1998 I had the opportunity to work towards my Diploma's degree under the guidance of Professor Kleinert. The result was a new algorithm for calculating highly accurate critical exponents from divergent perturbation expansions of field theories which I would like to summarize on this occasion. Typically, we possess L expansion coefficients of such a divergent series in powers of the bare coupling constant g_B, plus two more informations: The knowledge of the large-order behavior proportional to $(-\alpha)^k k! k^\beta g_B^k$, with a known growth parameter α, and the knowledge of the approach to scaling being of the type $c + c'/g_B^\omega$, with constants c, c' and a critical exponent of approach ω. The latter information leads to an increase in the speed of convergence and a high accuracy of the results. The algorithm is applied to the six- and seven-loop expansions for the critical exponents of $O(N)$-symmetric ϕ^4-theories in three dimensions, and the result for the critical exponent α is compared with a recent satellite experiment.

1 Introduction

The field-theoretic approach to critical phenomena provides us with power series expansions for the critical exponents of a wide variety of universality classes.

When inserted into the renormalization group equations, these expansions are supposed to determine the critical exponents via their values at an infrared-stable fixed point $g = g^*$. The latter step is nontrivial since the expansions are divergent and require a resummation, for which sophisticated

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methods have been developed [1]. The resummation methods use the information from the known large-order behavior $(-\alpha)^k k! k^\beta g_B^k$ of the expansions and analytic mapping techniques to obtain quite accurate results.

A completely different resummation procedure was developed on the basis of variational perturbation theory [2] to the expansions in powers of the *bare coupling constant*, which goes to infinity at the critical point.

This method converges as fast as the previous ones, even though it makes only use of the fact that the power series for the critical exponents approach their constant critical value in the form $c + c'/g_B^\omega$, where c, c' are constants, and ω is the critical exponent of the approach to scaling.

We may therefore expect that a resummation method which incorporates both informations should lead to results with an even higher accuracy, and it is our purpose to present such a method in the form of a simple algorithm [3].

2 The Problem

Mathematically, the problem we want to solve is the following: Let

$$f_L(g_B) = \sum_{k=0}^{L} f_k g_B^k \tag{1}$$

be the first L terms of a divergent asymptotic expansion

$$f(g_B) = \sum_{k=0}^{\infty} f_k g_B^k \tag{2}$$

with the large-order behavior of the expansion coefficients

$$f_k \stackrel{k\to\infty}{=} \gamma k! (-\alpha)^k k^\beta [1 + \mathcal{O}(1/k)]. \tag{3}$$

Suppose furthermore that $f(g_B)$, possesses a strong-coupling expansion of the type

$$f(g_B) = g_B^s \sum_{k=0}^{\infty} b_k g_B^{-k\omega}, \tag{4}$$

which is assumed to have some finite convergence radius $|g_B| \geq g_B^{\text{conv}}$.

We are interested in an efficient method to determine the strong-coupling coefficients b_k from the known coefficients f_k of the asymptotic expansion.

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3 Hyper-Borel Transformation

It turns out that ordinary Borel resummation is not well suited to solve this problem. Therefore we constructed an algorithm which is based on a generalization to be called *hyper-Borel transformation* [4] defined by

$$\tilde{B}(y) = \sum_{k=0}^{\infty} \tilde{B}_k y^k, \tag{5}$$

with coefficients

$$\tilde{B}_k \equiv \omega \frac{\Gamma\left(k(1/\omega - 1) + \beta_0\right)}{\Gamma\left(k/\omega - s/\omega\right)\Gamma(\beta_0)} f_k. \tag{6}$$

3.1 General Properties

The inverse of this transformation is given by the double integral

$$f(g_B) = \frac{\Gamma(\beta_0)}{2\pi i} \oint_C dt\, e^t t^{-\beta_0} \int_0^\infty \frac{dy}{y} \left[\frac{g_B}{y t^{(1-\omega)/\omega}}\right]^s \exp\left[\frac{y t^{(1-\omega)/\omega}}{g_B}\right]^\omega \tilde{B}(y), \tag{7}$$

as can easily be shown with the help of the integral representation of the inverse Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C dt e^t t^{-z}. \tag{8}$$

The transformation possesses a free parameter β_0 which can be used to optimize the approximation $f_L(g_B)$ at each order L. The power ω of the strong-coupling expansion is assumed to lie in the interval $0 < \omega < 1$, as it does in the upcoming physical applications.

The hyper-Borel transformation has the desired property of allowing for a resummation of $f_L(g_B)$ with the full sequence of powers of g_B in the strong-coupling expansion (4).

Our transform $\tilde{B}(y)$ shares with the ordinary Borel transform the property of being analytic at the origin and its radius of convergence is determined by the singularity on the negative real axis at

$$y_s = -\frac{1}{\sigma} \equiv -\frac{1}{\alpha} \frac{1}{\omega(1-\omega)^{1/\omega - 1}}. \tag{9}$$

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3.2 Resummation Procedure

A resummation procedure can now be set up on the basis of the transform $\tilde{B}(y)$. The inverse transformation (7) contains an integral over the entire positive axis, requiring an analytic continuation of the Taylor expansion of $\tilde{B}(y)$ beyond the convergence radius.

The reason for introducing the transform $\tilde{B}(y)$ was to allow us to reproduce the complete power sequence in the strong-coupling expansion (4), with a leading power g_B^s and a subleading sequence of powers $g_B^{s-k\omega}$, $k = 1, 2, 3, \ldots$. This is achieved by removing a factor $e^{-\rho\sigma y}$ with $\rho, \sigma > 0$ from the truncated series (5) of our transform $\tilde{B}(y)$. Furthermore, by removing a second simple prefactor of the form $(1 + \sigma y)^{-\delta}$, we weaken the leading singularity in the hyper-Borel complex y-plane, which determines the large order behavior (3). The remaining series has still a finite radius of convergence. To achieve convergence on the entire positive y-axis for which we must do the integral (7), we reexpand the remaining series of y in powers of the conformal mapping $\kappa(y)$ given by

$$\kappa(y) = \frac{\sigma y}{1 + \sigma y}, \tag{10}$$

which maps a shifted right half of the complex y-plane with $\Re[y] \geq -1/2\sigma$ onto the unit circle in the complex κ-plane. Thus we reexpand $\tilde{B}(y)$ in the following way:

$$\tilde{B}(y) \equiv \sum_{k=0}^{\infty} \tilde{B}_k y^k = e^{-\rho\sigma y}[1 + \sigma y]^{-\delta} \sum_{k=0}^{\infty} h_k \, \kappa^k(y) = e^{-\rho\sigma y} \sum_{k=0}^{\infty} h_k \frac{(\sigma y)^k}{(1 + \sigma y)^{k+\delta}}. \tag{11}$$

The inverse hyper-Borel transform of $\tilde{B}(y)$ is now found by forming the integrals of the expansion functions in (11)

$$I_n(g_B) = \frac{\Gamma(\beta_0)}{2\pi i} \oint_C dt e^t t^{-\beta_0} \times \int_0^\infty \frac{dy}{y} \left[\frac{g_B}{y t^{1/\omega - 1}}\right]^s \exp\left[-\frac{y t^{1/\omega - 1}}{g_B}\right]^\omega e^{-\rho\sigma y} \frac{(\sigma y)^n}{(1 + \sigma y)^{n+\delta}}, \tag{12}$$

so that the approximants $f_L^a(g_B)$ may be written as

$$f_L^a(g_B) = \sum_{n=0}^{L} h_n I_n(g_B). \tag{13}$$

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The convergent strong-coupling expansion of $I_n(g_B)$ is obtained by performing a Taylor series expansion of the exponential function in (12), which is an expansion in powers of $1/g_B^\omega$. After integrating over t and y we obtain an expansion

$$I_n(g_B) = g_B^s \sum_{k=0}^{\infty} b_k^{(n)} g_B^{-k\omega}, \tag{14}$$

which has indeed the same power sequence as the strong-coupling expansion (4) of the function $f(g_B)$ to be resummed and the expansion coefficients are given by

$$b_k^{(n)} = \frac{(-1)^k}{k!} \frac{\sigma^{s-k\omega}\Gamma(\beta_0)}{\Gamma[(\omega-1)k+\beta_0+(1/\omega-1)s]} i_k^{(n)}, \tag{15}$$

where $i_k^{(n)}$ denotes the integral

$$i_k^{(n)} = \int_0^\infty dy e^{-\rho y}(1+y)^{-\delta-n} y^{k\omega+n-s-1}. \tag{16}$$

For large k, the integral on the right-hand side of (16) can be estimated with the help of the saddle-point approximation, which shows that the strong-coupling expansion (4) has a finite convergence radius

$$|g_B| \geq \frac{(\rho\sigma)^\omega}{(1-\omega)^{1-\omega}}, \tag{17}$$

implying that the basis functions $I_n(g_B)$, and certainly also $f(g_B)$ itself, possess additional singularities beside $g_B = 0$. The parameter ρ will be optimally adjusted to match the positions of these singularities.

3.3 Convergence Properties of Resummed Series

We shall now discuss the speed of convergence of the resummation procedure. For this it will be sufficient to estimate the convergence of the strong-coupling coefficients b_k^L of the approximations $f_L(g_B)$ against the true b_k in (4). The convergence for arbitrary values of g_B will always be better than that. Such an estimate is possible by looking at the large-n behavior of the expansion coefficients $b_k^{(n)}$ in the strong-coupling expansion of $I_n(g_B)$ in (14). This is determined by the saddle-point approximation to the integral $i_k^{(n)}$ in Eq. (16),

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which we rewrite as

$$i_k^{(n)} = \int_0^\infty dy e^{-\rho y - n \ln(1+1/y)} (1+y)^{-\delta} y^{k\omega - s - 1}. \tag{18}$$

The saddle point lies at

$$y_s = \sqrt{\frac{n}{\rho}} \left[1 + \mathcal{O}(1/\sqrt{n})\right]. \tag{19}$$

At this point, the total exponent in the integrand is

$$-\rho y_s - n \ln\left(1 + \frac{1}{y_s}\right) = -2\sqrt{\rho n} \left[1 + \mathcal{O}(1/\sqrt{n})\right], \tag{20}$$

implying the large-n behavior

$$b_k^{(n)} \overset{n\to\infty}{=} \text{const.} \times n^{k\omega - s - 1 - \delta} e^{-2\sqrt{\rho n}} \left[1 + \mathcal{O}(1/\sqrt{n})\right]. \tag{21}$$

The coefficients b_k^L of the approximations $f_L^a(g_B)$ are linear combinations of the coefficients $b_k^{(n)}$ of the basis functions $I_n(g_B)$:

$$b_k^L = \sum_{n=0}^{L} b_k^{(n)} h_n. \tag{22}$$

The speed of convergence with which the b_k^L's approach b_k as the number L goes to infinity is governed by the growth with n of the reexpansion coefficients h_n and of the coefficients $b_k^{(n)}$ in Eq. (21). We shall see that for the series to be resummed, the reexpansion coefficients h_n will grow at most like some power n^r, implying that the approximations b_k^L approach their $L \to \infty$ -limit b_k with an error proportional to

$$b_k^L - b_k \sim L^{r + k\omega - s - \delta - 1/2} \times e^{-2\sqrt{\rho L}}. \tag{23}$$

This is the important advantage of the present resummation method with respect to variational perturbation theory [2,5], where the error decreases merely like $e^{-\text{const} \times L^{1-\omega}}$ with $1 - \omega$ close to $1/4$.

4 Resummation of Ground-State Energy of the Anharmonic Oscillator

Before beginning with the resummation of the perturbation expansions for the critical exponents of ϕ^4-field theories, it will be useful to obtain a feeling for the quality of the above-developed resummation procedure, in particular

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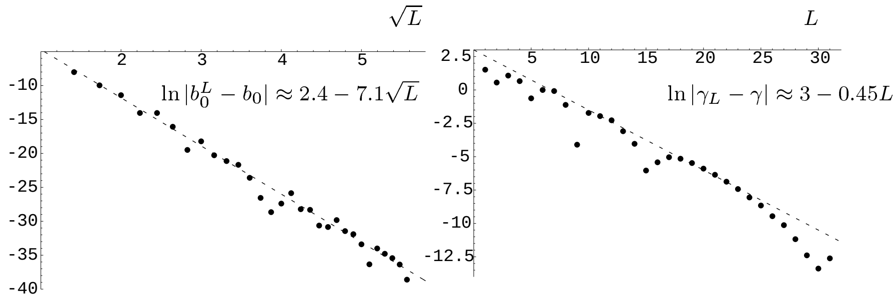

Figure 1. Logarithmic plot of the convergence behavior of the successive approximations to the prefactor γ in the large-order behavior (3), and of the leading strong-coupling coefficient b_0^L.

for the significance of the parameters upon the speed of convergence. We do this by resumming the often-used example of an asymptotic series, the perturbation expansion of the ground-state energy of the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + m^2\frac{x^2}{2} + gx^4. \tag{24}$$

The ground state of this quantum mechanical system has an asymptotic expansion with a large-order behavior of the form (3), where the growth parameters are given by $\alpha = 3, \beta = -1/2, \gamma = \sqrt{6/\pi^3}$, and it also possesses a strong-coupling expansion (4) with the parameters $s = 1/3$, $\omega = 2/3$. We fix the parameters in our resummation method by the condition that the approximants $E_0^L(g)$ of the ground-state energy $E_0(g)$ obey Eqs. (3) and (4) with the same parameters as $E_0(g)$. The best choice of β_0 will be made differently depending on the regions of g.

Let us test the convergence of our algorithm at small negative coupling constants g, i.e. near the tip of the left-hand cut in the complex g-plane. We do this by calculating the prefactor γ in the large-order behavior (3). In this case the convergence turns out to be fastest by giving the parameter β_0 a small value, i.e. $\beta_0 = 2$.

The values of the approximants γ_L are shown in Fig. 1. They converge exponentially fast against the exact limiting value. The convergence of the strong-coupling coefficients b_k^L is given by the stretched exponential $\approx e^{-\text{const}\times\sqrt{L}}$ [see Eq. (23)], rather than $\approx e^{-\text{const}\times L^{1/3}}$ for variational perturbation theory. The latter is seen on the right-hand side of Fig. 1.

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Table 1. Strong-coupling coefficients b_n of the 70-th order approximants $E^0_{70}(g) = \sum_{n=0}^{70} h_n I_n(g)$ to the ground-state energy $E^0(g)$ of the anharmonic oscillator. They have the same accuracy as the variational perturbative calculations up to order 251 in Refs. [2,6].

n	b_n
0	0.667 986 259 155 777 108 270 962 02
1	0.143 668 783 380 864 910 020 319
2	−0.008 627 565 680 802 279 127 963
3	0.000 818 208 905 756 349 542 41
4	−0.000 082 429 217 130 077 219 91
5	0.000 008 069 494 235 040 964 75
6	−0.000 000 727 977 005 945 772 63
7	0.000 000 056 145 997 222 351 17
8	−0.000 000 002 949 562 732 709 36
9	−0.000 000 000 064 215 331 956 97
10	0.000 000 000 048 214 263 789 07

We have applied our resummation method to the first 10 strong-coupling coefficients using the expansion coefficients f_k up to order 70. The results are shown in Table 1. Comparison with a similar table in Refs. [2,6] shows that the new resummation method yields in 70th order the same accuracy as variational perturbation theory did in 251st order.

5 Resummation of Critical Exponents

Having convinced ourselves of the fast convergence of our new resummation method, let us now turn to the perturbation expansions of the $O(N)$-symmetric ϕ^4-theories in powers of the bare coupling constant λ_B in $D = 3$ dimensions. If we introduce the dimensionless bare coupling constant $g_B \equiv \lambda_B/m$, where m is the renormalized mass, the critical exponents are defined by

$$\eta = g_B \frac{d}{dg_B} \log Z_\Phi \Big|_{g_B=\infty},$$
$$2 - \nu^{-1} = g_B \frac{d}{dg_B} \log \frac{m_0^2}{m^2} \Big|_{g_B=\infty}. \tag{25}$$

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Table 2. Critical exponents of the $O(N)$-symmetric ϕ^4-theory from our new resummation method.

n	γ	η	ν	ω
0	1.1604[8] (4){0.075}	0.0285[6](4){0.037}	0.5881[8](4){0.075}	0.803[3]{1}
1	1.2403[8] (4){0.110}	0.0335[6](3){0.043}	0.6303[8](4){0.065}	0.792[3]{1}
2	1.3164[8] (5){0.033}	0.0349[8](5){0.042}	0.6704[7](4){0.098}	0.784[3]{1}
3	1.3882[10](7){0.210}	0.0350[8](5){0.043}	0.7062[7](4){0.110}	0.783[3]{1}

The expansions of the field renormalization constant Z_Φ and the bare mass m_0 have been calculated up to seventh order in g_B in the literature [7]. When approaching the critical point, the renormalized mass m tends to zero, so that the problem is to find the strong-coupling limit $g_B \to \infty$ of these expansions. In order to have the critical exponents approach a constant value, the power s in Eq. (4) must be set equal to zero.

In contrast to the quantum mechanical discussion in the last section, the exponent ω governing the approach to the scaling limit is now unknown, and must also be determined from the available perturbation expansions. As in Ref. [5,8], we solve this problem by using the fact that the existence of a critical point implies the renormalized coupling constant g in powers of g_B to converge against a constant renormalized coupling g^* for $m \to 0$.

The convergence against a fixed coupling g^* occurs only for the correct value of ω in the resummation functions $I_n(g_B, \omega)$. At different values, $g(g_B)$ has some strong-coupling power behavior g_B^s with $s \neq 0$. We may therefore determine ω by forming a series for the power s,

$$s = \frac{d \log g(g_B)}{d \log g_B} = \frac{g_B}{g} g'(g_B), \tag{26}$$

resumming this for various values of ω in the basis functions, and finding the critical exponent ω from the zero of s. Alternatively, since $g(g_B) \to g^*$, we can just as well resum the series for $-gs$, which coincides with the β-function of renormalization group theory [not to be confused with the growth parameter β in (3)]

$$\beta(g_B) \equiv -g_B \frac{dg(g_B)}{dg_B}. \tag{27}$$

The results for the critical exponents of all $O(N)$-symmetries are shown in Table 2. The total error is indicated in the square brackets. It is deduced

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from the error of resummation of the critical exponent at a fixed value of ω indicated in the parentheses, and from the error $\Delta\omega$ of ω, using the derivative of the exponent with respect to ω given in curly brackets. Symbolically, the relation between these errors is

$$[\ldots] = (\ldots) + \Delta\omega\{\ldots\}. \tag{28}$$

The accuracy of our results can be judged by comparison with the most accurately measured critical exponent α parameterizing the divergence of the specific heat of superfluid helium at the λ-transition by $|T_c - T|^{-\alpha}$. By going into a vicinity of the critical temperature with $\Delta T \approx 10^{-8}$ K, a recent satellite experiment has provided us with the value [9] $\alpha = -0.01056 \pm 0.00038$. Our value for α is deduced from ν in Table 2 via the hyper-scaling relation $\alpha = 2 - 3\nu$ to $\alpha = -0.0112 \pm 0.0021$, in good agreement with the experimental number.

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