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VARIATIONAL PERTURBATION THEORY: A POWERFUL METHOD FOR DERIVING STRONG-COUPLING EXPANSIONS

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In this contribution, an overview of Kleinert's variational perturbation theory will be given. Contrary to standard perturbative approaches, this method yields converging approximations uniformly in the coupling strength of anharmonic terms. In standard quantum mechanics, the simplest example is the one-dimensional anharmonic oscillator. It is shown how variational perturbation theory can be exploited to calculate the convergent *strong*-coupling expansion from the divergent *weak*-coupling perturbation series. By recalling the Duru-Kleinert path-integral solution of the three-dimensional hydrogen problem, I make use of the mapping between Coulomb systems in three dimensions and oscillator systems in four dimensions to also derive the quantum mechanical strong-coupling expansion of the ground-state energy of the hydrogen problem allowing for an isotropic gr-perturbation.

1 Introduction

Most perturbation expansions in physics are divergent asymptotic series whose large-order coefficients grow factorially. Typical examples include the perturbative calculation of the anomalous magnetic moment of the electron in quantum electrodynamics, field theoretical ϵ-expansions of critical exponents in statistical physics, and the low-field expansions of the Stark and Zeeman effects in atomic physics.

The study of the large-order behavior of such asymptotic series and the problem of developing appropriate resummation techniques has a long history in Hagen Kleinert's research group. In fact, shortly after having finished my

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diploma thesis under his guidance and having jointly solved the "particle-in-a-box" problem with path-integral techniques [1], this was one of the first research topics we started working on together. After an intense initial period we returned to this problem many times [2–4] and collected over the years quite an extensive set of lecture notes [5]. While working on systematic improvements of the variational approximation [6,7] underlying an effective classical description of quantum statistics, Kleinert [8] realized in 1993 that this could lead to a powerful resummation scheme for divergent perturbation series. Remembering our joint work on the variational scheme [9,10], here our interests met again and we began to work [11–13] on what will be briefly described in the following.

The paradigm for a divergent asymptotic perturbation series expansion is the ground-state energy $E^{(0)}(g)$ of an anharmonic oscillator with potential $V(x) = \omega^2 x^2/2 + g x^4/4$ $(\omega^2, g > 0)$. The weak-coupling Rayleigh-Schrödinger perturbation series takes the form

$$E^{(0)}(g) = \omega \sum_{k=0}^{\infty} E_k^{(0)} \left(\frac{g/4}{\omega^3}\right)^k , \tag{1}$$

where the coefficients $E_k^{(0)} = 1/2,\ 3/4,\ -21/8,\ 333/16,\ -30885/128, \ldots$ can be shown to grow asymptotically as [14]

$$E_k^{(0)} = -(1/\pi)(6/\pi)^{1/2}(-3)^k k^{-1/2} k! \left[1 + \mathcal{O}(1/k)\right] . \tag{2}$$

Only for very weak couplings g, a direct evaluation of the power-series expansion truncated at a finite order $k \propto 1/g$ can yield a reasonably good approximation [5] with an error of the order of $\exp(-\text{const}/g)$. At larger couplings the partial sums become very erratic and hence completely useless, unless some resummation procedure is applied. The accuracy of standard techniques such as Padé or Borel resummation [5] also deteriorates, however, quite rapidly in the strong-coupling limit, where $E^{(0)}(g)$ has an expansion of the form

$$E^{(0)}(g) = (g/4)^{1/3} \left[\alpha_0 + \alpha_1 (4\omega^3/g)^{2/3} + \alpha_2 (4\omega^3/g)^{4/3} + \ldots\right] . \tag{3}$$

2 Variational Perturbation Theory

Variational perturbation theory [7,8,15], on the other hand, yields a sequence of *exponentially* fast converging approximations *uniformly* in g (Refs. [11–

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13,16]), and thus provides a powerful method for calculating the strong-coupling expansion coefficients α_i in (3). As mentioned in the historical remarks above, the starting point of this approach was the variational principle for evaluating quantum partition functions in the path-integral formulation [6,7]. While in many applications the accuracy was found to be excellent over a wide range of temperatures [10], slight deviations from exact [6,7,9,10] or computer simulation [17] results at very low temperatures motivated a systematic study of higher-order corrections [7,8,15].

In the zero-temperature limit the higher-order calculations simplify and lead to a resummation scheme for the energy eigenvalues which can be summarized as follows. First, the harmonic term of the potential is decomposed into $\omega^2 x^2 = \Omega^2 x^2 + \left(\omega^2 - \Omega^2\right) x^2$, where Ω is a trial frequency to be optimized later, and the potential is rewritten as $V(x) = \Omega^2 x^2/2 + g(-2\sigma x^2/\Omega + x^4)/4$, with $\sigma = \Omega(\Omega^2 - \omega^2)/g$. Keeping σ fixed, one then performs a perturbation expansion of $\hat{E}_N^{(0)} \equiv E_N^{(0)}/\Omega$ in powers of $\hat{g} \equiv g/\Omega^3$,

$$\hat{E}_N^{(0)}(\hat{g},\sigma) = \sum_{k=0}^{N} \varepsilon_k^{(0)}(\sigma)\,(\hat{g}/4)^k\,, \tag{4}$$

$$\varepsilon_k^{(0)}(\sigma) = \sum_{j=0}^{k} E_j^{(0)} \binom{(1-3j)/2}{k-j} (-4\sigma)^{k-j}, \tag{5}$$

which can be readily derived by inserting $\omega = (\Omega^2 - g\sigma/\Omega)^{1/2} = \Omega(1-\hat{g}\sigma)^{1/2}$ into the original perturbation series (1) and reexpanding in powers of $\hat{g}$. By construction the truncated power series $W_N(g,\Omega) \equiv \Omega \hat{E}_N^{(0)}(\hat{g},\sigma)$ becomes independent of Ω in the limit $N \to \infty$. At any finite order, however, it *does* depend on Ω, the approximation having its fastest speed of convergence where it depends least on Ω, i.e. at points where $\partial W_N/\partial\Omega = 0$. If we denote the order-dependent optimal value of Ω by Ω_N, the quantity $W_N(g,\Omega_N)$ is the new approximation to $E^{(0)}(g)$.

While W_N is a polynomial in Ω of degree $3N$ with g-dependent coefficients, it can be proven [11] that its derivative with respect to Ω admits a compact, factorized representation, namely $\partial W_N/\partial\Omega = (\hat{g}/4)^N P_N(\sigma)$, where $P_N(\sigma) = -2d\varepsilon_{N+1}^{(0)}(\sigma)/d\sigma$ is a polynomial of degree N in σ. This observation simplifies the calculations considerably and shows that the optimal solutions Ω_N depend only trivially on g through $\sigma_N = \Omega_N(\Omega_N^2 - \omega^2)/g$. The order-dependent

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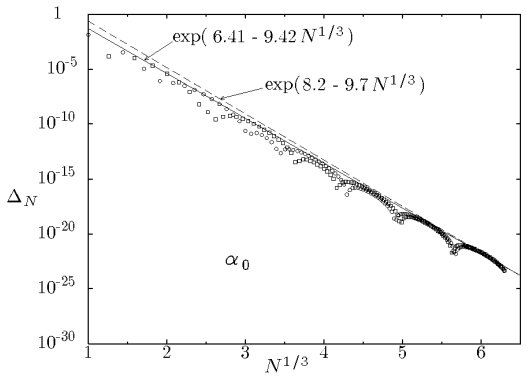

Figure 1. Exponential accuracy as shown by the rapidly decreasing error $\Delta_N = |\alpha_0^{(N)} - \alpha_0|$ of the Nth approximant (7) for the strong-coupling coefficient α_0, where $N = 1, \ldots, 251$ is the order of the weak-coupling expansion.

optimal values of σ are found to be well fitted by

$$\sigma_N = cN\left(1 + 6.85/N^{2/3}\right), \tag{6}$$

with $c = 0.186\,047\,272\ldots$ determined analytically (cf. Section 3). This observation suggested [12] that the variational resummation scheme can be taken directly to the strong-coupling limit by introducing the reduced frequency $\hat{\omega} = \omega/\Omega$, rewriting the approximation as $W_N = (g/\hat{g})^{1/3}\, w_N(\hat{g}, \hat{\omega}^2)$, and expanding the function $w_N(\hat{g}, \hat{\omega}^2)$ in powers of $\hat{\omega}^2 = (\omega^3/g)^{2/3}\hat{g}^{2/3}$. As a result, for $E^{(0)}(g) \approx W_N(g)$ we find an expansion of the form (3) with coefficients [12]

$$\alpha_n^{(N)} = (\hat{g}/4)^{(2n-1)/3} \sum_{k=0}^{N} (-1)^{k+n} \sum_{j=0}^{k-n} E_j^{(0)} \binom{(1-3j)/2}{k-j} \binom{k-j}{n} (-\hat{g}/4)^j. \tag{7}$$

If this is evaluated at $\hat{g} = 1/\sigma_N$ with σ_N given in (6), we obtain the exponentially fast approach to the exact limit as shown in Fig. 1 for α_0. Notice the oscillatory modulations. The computation of the higher-order coefficients α_n proceeds similarly and the results up to $n = 22$ can be found in Table 1 of Ref. [12]. Our results for α_0 and α_1 based on the first 251 weak-coupling expansion coefficients are compared in Table 1 with other recent estimates. So far the more mathematically motivated resummation scheme of Ref. [18]

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Table 1. Estimates of the leading strong-coupling expansion coefficients α_n.

α_0	Ref.
0.667 986 259 155 777 108 270 96	[12]
0.667 986 259 155 777 108 270 962 016 919 86	[18]
0.667 986 259 155 777 108 270 962 016 919 860 199 4	[19]
0.667 986 259 155 777 108 270 962 016 919 860 199 430 404 936 984 ...	[21]
α_1	Ref.
0.143 668 783 380 864 910 020 3	[12]
0.143 668 783 380 864 910 020 319 127 583 17	[18]
0.143 668 783 380 864 910 020 319 127 583 168 634 2	[19]

gives more accurate numbers than those in Ref. [12]. However, by applying extrapolation techniques to the sequences for α_n [19], the accuracy of our estimates can be further considerably improved. Specifically we employed Wynn's ϵ-algorithm [20] where the extrapolants $\epsilon_k^{(n)}$ are defined recursively by $\epsilon_k^{(-1)} = 0$, $\epsilon_k^{(0)} = W_k$, and $\epsilon_k^{(n+1)} = \epsilon_{k+1}^{(n-1)} + 1/(\epsilon_{k+1}^{(n)} - \epsilon_k^{(n)})$.

3 Convergence Behavior

It is well known that the ground-state energy $E^{(0)}(g)$ of the quartic anharmonic oscillator satisfies a subtracted dispersion relation which implies an integral representation for the perturbation coefficients [14],

$$E_k^{(0)} = \frac{4^k}{2\pi i} \int_0^{-\infty} \frac{dg}{g^{k+1}} \mathrm{disc} E^{(0)}(g), \tag{8}$$

where $\mathrm{disc}\, E^{(0)}(g) = 2i\mathrm{Im}\, E^{(0)}(g - i\eta)$ denotes the discontinuity across the left-hand cut in the complex g-plane. For large k, only its $g \to 0^-$ behavior is relevant and a semiclassical calculation yields $\mathrm{disc}\, E^{(0)}(g) \approx -2i\omega(6/\pi)^{1/2}(-4\omega^3/3g)^{1/2} \exp(4\omega^3/3g)$, which in turn implies the large-order behavior (2) of $E_k^{(0)}$.

The reexpanded series (5) is obtained from (1) by replacing $\omega \to \Omega(1 - \sigma\hat{g})^{1/2}$, which, in terms of the coupling constant, amounts to $\bar{g} \equiv g/\omega^3 \to \hat{g}/(1 - \sigma\hat{g})^{3/2}$. This implies [13] a dispersion relation for $\hat{E}^{(0)} \equiv E^{(0)}/\Omega$ and

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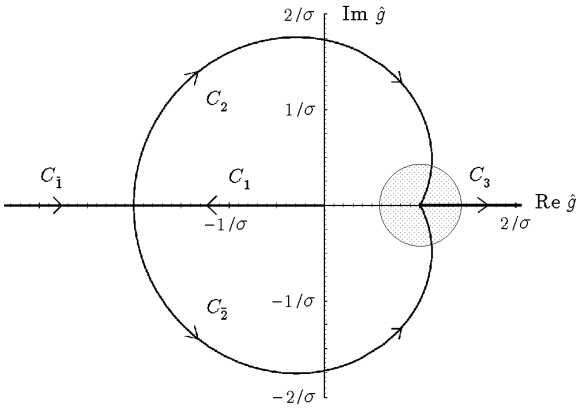

Figure 2. Cuts in the complex $\hat{g}$-plane. The shaded area shows the circle of convergence of the strong-coupling expansion.

hence the coefficients $\varepsilon_k^{(0)}$,

$$\varepsilon_k^{(0)} = \frac{4^k}{2\pi i}\int_C \frac{d\hat{g}}{\hat{g}^{k+1}} \mathrm{disc}_C \hat{E}^{(0)}(\hat{g}), \tag{9}$$

where $\mathrm{disc}_C \hat{E}^{(0)}(\hat{g})$ is the discontinuity across the cuts C in the complex $\hat{g}$-plane shown in Fig. 2. The cuts C run along the contours $C_1, C_{\bar{1}}, C_2, C_{\bar{2}}$, the images of the left-hand cut in the complex g-plane, and C_3, originating from the square root of $1-\sigma\hat{g}$ in the mapping from $\bar{g}$ to $\hat{g}$.

Let us now discuss the contributions of the various cuts to the Nth term S_N. For the cut C_1 and the empirically observed optimal solutions $\sigma_N = cN(1+b/N^{2/3})$, a saddle-point approximation shows [13] that this term gives a convergent contribution, $S_N(C_1) \propto e^{-[-b\log(-\gamma)+(cg)^{-2/3}]N^{1/3}}$, if one chooses $c = 0.186\,047\,272\ldots$ and $\gamma = -0.242\,964\,029\ldots$. Inserting the fitted value of $b = 6.85$, this yields an exponent of $-b\log(-\gamma) = 9.7$, in rough agreement with the convergence seen in Fig. 1. If this would be the only contribution, the convergence behavior could be changed at will by varying the parameter b. For $b < 6.85$, a slower convergence is indeed observed. The convergence cannot be improved, however, by choosing $b > 6.85$, since the optimal convergence is limited by the contributions of the other cuts.

The cut $C_{\bar{1}}$ is still harmless; it contributes a last term $S_N(C_{\bar{1}})$ of the negligible order $e^{-N\log N}$. The cuts $C_{2,\bar{2},3}$, however, deserve a careful con-

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sideration. If they would really start at $\sigma\hat{g} = 1$, the leading behavior would be $\varepsilon_k^{(0)}(C_{2,\bar{2},3}) \propto \sigma^k$, and therefore $S_N(C_{2,\bar{2},3}) \propto (\sigma\hat{g})^N$, which would be in contradiction to the empirically observed convergence in the strong-coupling limit. The important point is that the cuts in Fig. 2 do not really reach the point $\sigma\hat{g} = 1$. There exists a small circle of radius $\Delta\hat{g} > 0$ in which $\hat{E}^{(0)}(\hat{g})$ has no singularities at all, a consequence of the fact that the strong-coupling expansion (3) converges for $g > g_s$. The complex conjugate pair of singularities gives a contribution $S_N(C_{2,\bar{2},3}) \approx e^{-N^{1/3}a\cos\theta}\cos(N^{1/3}a\sin\theta)$, with $a = 1/(|\bar{g}_s|c)^{2/3}$. By analyzing the convergence behavior of the strong-coupling series we find $|\bar{g}_s| \approx 0.160$ and $\theta \approx -0.467$, which implies an asymptotic falloff of $e^{-9.23N^{1/3}}$ for the envelope, and furthermore also explains the oscillations in the data [13].

4 Three-Dimensional Coulomb Systems

It is well known that Coulomb systems in *three* dimensions (3D) and oscillator systems in *four* dimensions (4D) are closely related to each other [22,23]. This property has been exploited in many ways; in particular it was one of the clues for the solution of the path-integral for the hydrogen atom by Duru and Kleinert [23]. One recent example is a simplified analysis of the large-order behavior of weak-coupling expansions for perturbed Coulomb systems [2,3]. Here we shall concentrate on strong-coupling expansions which are usually more difficult to derive. Specifically we consider the 3D Coulomb system with the Hamiltonian

$$H_C = \frac{1}{2}\vec{p}^2 - \frac{1}{r} + gr, \tag{10}$$

which is related to the 4D anharmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 + \frac{\omega^2}{2}\mathbf{x}^2 + \lambda(\mathbf{x}^2)^2. \tag{11}$$

In particular the ground-state energies of the two systems can be mapped onto each other. If E denotes the ground-state energy of H_C and ϵ is the ground-state energy of H, then the relation reads [2]

$$\epsilon = \epsilon(\omega, \lambda) = 1, \tag{12}$$

with

$$\omega^2 = -E/2, \qquad \lambda = g/16. \tag{13}$$

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By simple scaling arguments the energy E of the Coulomb systems is easily seen to possess a weak-coupling series expansion in powers of g,

$$E = E_0 + E_1 g + E_2 g^2 + \ldots, \tag{14}$$

and a strong-coupling expansion of the form

$$E = g^{2/3} \left[a_0 + a_1 g^{-1/3} + a_2 g^{-2/3} + \ldots \right]. \tag{15}$$

Similarly, for the 4D anharmonic oscillator the weak-coupling expansion reads

$$\epsilon = \omega \left[\epsilon_0 + \epsilon_1 (\lambda/\omega^3) + \epsilon_2 (\lambda/\omega^3)^2 + \ldots \right], \tag{16}$$

and the strong-coupling expansion takes the form

$$\epsilon = \lambda^{1/3} \left[\alpha_0 + \alpha_1 (\lambda/\omega^3)^{-2/3} + \alpha_2 (\lambda/\omega^3)^{-4/3} + \ldots \right]. \tag{17}$$

4.1 4D Anharmonic Oscillator Strong-Coupling Expansion

To calculate the strong-coupling coefficients a_n in (15) by using the relations (10)-(13) we first have to derive the strong-coupling expansion of the 4D anharmonic oscillator. Here this task is accomplished by means of variational perturbation theory as described in the previous section for the one-dimensional case. In four dimensions the necessary input information, which is the weak-coupling perturbation coefficients ϵ_i, can also be easily generated to very high orders by applying recursion relations first discussed by Bender and Wu [14]. The present calculation is based on an expansion up to order 290. The resummation scheme of variational perturbation theory then yields again exponentially fast convergent sequences $\alpha_n^{(N)}$ for the strong-coupling coefficients α_n and, as discussed above, their accuracy may be further improved by applying standard extrapolation techniques such as Wynn's ϵ-algorithm [20]. This procedure was applied [24] to obtain the expansion coefficients shown in Table 2.

4.2 3D Coulomb Strong-Coupling Expansion

By making use of the relations (12) and (13) it is straightforward to express the expansion coefficients for the Coulomb system in terms of the expansion coefficients for the anharmonic oscillator. For the mapping between the weak-coupling series see, e.g., Ref. [2]. Similarly, by inserting the strong-coupling

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Table 2. Coefficients α_n of the strong-coupling expansion (17) for the ground-state energy of the 4D anharmonic oscillator (11). The symbol # denotes the number of digits obtained.

n	α_n	#
0	3.398 150 176 027 696 746 352 787 969 422 624 006 593 241 57	45
1	0.447 038 467 415 823 402 400 410 319 616 607 612 580 077 2	43
2	−0.015 633 102 347 011 889 354 006 985 272 609 625 470 865	42
3	0.000 806 409 491 306 496 503 927 969 548 053 909 647 16	41
4	−0.000 039 561 514 296 026 965 992 526 411 214 682 179 66	41
5	0.000 001 484 265 174 534 240 244 510 299 896 097 807 89	41
6	−0.000 000 013 262 160 340 168 018 805 647 156 427 568 43	41
7	−0.000 000 004 230 227 654 282 595 813 731 333 132 587 68	41
8	0.000 000 000 479 462 248 736 079 997 207 517 503 895 6	40
9	−0.000 000 000 029 933 252 179 913 943 227 242 901 209 2	40
10	0.000 000 000 000 777 862 757 469 125 859 005 974 522 4	40
...	...	...
50	$\approx 6 \times 10^{-56}$	57

expansion (17) into (12) and using (13) to replace ω and λ by E and g, we obtain

$$\hat{\alpha}_0 + \hat{\alpha}_1 \frac{E}{g^{2/3}} + \hat{\alpha}_2 \left(\frac{E}{g^{2/3}}\right)^2 + \cdots = \frac{16^{1/3}}{g^{1/3}}, \tag{18}$$

where

$$\hat{\alpha}_i \equiv \left(-\frac{16^{2/3}}{2}\right)^i \alpha_i \tag{19}$$

are rescaled strong-coupling coefficients of the anharmonic oscillator. Next we insert the strong-coupling expansion (15) of E and equate equal powers of $g^{-1/3}$. By defining the auxiliary sums

$$S_k \equiv \sum_{n=0}^{k} \binom{n}{k} \hat{\alpha}_n a_0^{n-k}, \tag{20}$$

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Table 3. Coefficients a_n of the strong-coupling expansion (15) for the ground-state energy of the 3D Coulomb system (10) with gr-perturbation. The symbol # denotes the number of digits obtained.

n	a_n	#
0	1.855 757 081 489 238 479	19
1	−1.051 866 501 087 132 24	18
2	−0.186 184 039 301 421 2	16
3	−0.036 881 399 705 264	15
4	−0.003 821 866 269 00	14
5	0.000 631 796 785	12
6	0.000 303 537 651	12
7	0.000 008 501 572	12
8	−0.000 022 637 496	12
9	−0.000 004 975 49	11
10	0.000 001 184 37	11

where $\binom{n}{k}$ denotes the standard binomial coefficient, this leads to the equation

$$S_0 = 0, \tag{21}$$

which determines a_0 as one of the roots of S_0, and in addition to a set of recursive relations for the higher coefficients a_k, with $k > 0$,

$$\begin{aligned}
a_1 &= 16^{1/3}/S_1, \\
a_2 &= -a_1^2 S_2/S_1, \\
a_3 &= -(2a_1a_2S_2 + a_1^3S_3)/S_1, \\
a_4 &= -\left[(a_2^2 + 2a_1a_3)S_2 + 3a_1^2a_2S_3 + a_1^4S_4\right]/S_1, \\
a_5 &= -\left[2(a_1a_4 + a_2a_3)S_2 + 3(a_1^2a_3 + a_2^2a_1)S_3 + 4a_1^3a_2S_4 + a_1^5S_5\right]/S_1,
\end{aligned} \tag{22}$$

and so on.

By inserting the strong-coupling expansion coefficients for the anharmonic oscillator into Eq. (20), compiled in Table 2, it is now straightforward to solve (21) for a_0 and then to evaluate the explicit expressions (22). The results up to the 10th order in $g^{-1/3}$ are collected in Table 3. By using a different method, Fernández [25] obtained: $a_0 = 1.855\,757\,081\,49$, $a_1 = -1.051\,866\,501\,1$, $a_2 =$

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$-0.186\,184\,039\,3$, $a_3 = -0.036\,881\,399\,7$, and $a_4 = -0.003\,821\,866\,3$.

4.3 Expansion around g_0 with $E(g_0) = 0$

Another solution of (18) is obviously given by

$$E = 0, \tag{23}$$

$$g_0 = 16/\alpha_0^3 = 0.407\,748\ldots. \tag{24}$$

Here $\alpha_0 = 3.398\,150\ldots$ was used from Table 2. The numerical value (24) is thus the coupling constant for which the ground-state energy of the perturbed Coulomb problem (10) assumes the special value zero. This may be used as the starting point for a systematic expansion of the form

$$E = b_1(g/g_0 - 1) + b_2(g/g_0 - 1)^2 + \ldots, \tag{25}$$

with

$$b_1 = -\frac{1}{3}\frac{\hat{\alpha}_0}{\hat{\alpha}_1} g_0^{2/3}, \tag{26}$$

$$b_2 = -\frac{\hat{\alpha}_2}{\hat{\alpha}_1}\frac{b_1^2}{g_0^{2/3}}, \tag{27}$$

and so on. Using the numbers for α_n given in Table 2 this leads to $b_1 = 0.438\,855\ldots$ and $b_2 = -0.038\,886\ldots$. A comparison of this "$(g - g_0)$"-expansion with the strong-coupling series is shown in Fig. 3. Notice the very good agreement even for relatively small coupling constants g.

5 Conclusions

Variational perturbation theory is a perfect tool for converting the divergent weak-coupling perturbation series of anharmonic systems into a sequence of exponentially fast converging approximations for the strong-coupling expansion. For the anharmonic oscillator, the empirically observed convergence behavior with superimposed oscillations can be explained by identifying the relevant singularities in a dispersion integral representation. A combination of variational perturbation theory with the mapping of 3D Coulomb onto 4D oscillator systems allows to compute with high accuracy the strong-coupling expansion for the ground-state energy E of a perturbed Coulomb system. As a by-product of this approach, a novel expansion around the coupling constant g_0 defined by $E(g_0) = 0$ can also be derived.

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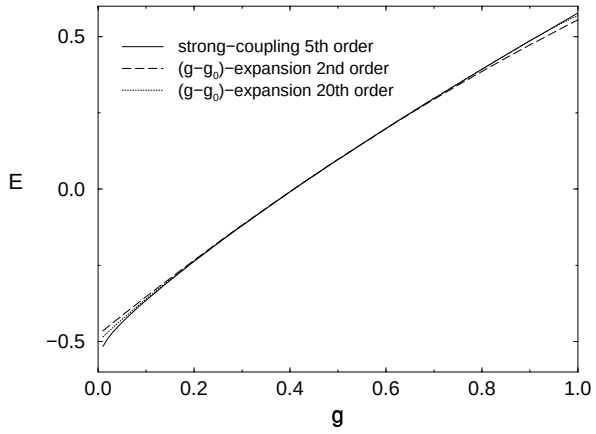

Figure 3. Comparison of strong-coupling and $(g - g_0)$-expansions for the ground-state energy E of the Coulomb system (10).

In order to focus on the main points, I have here illustrated the basic ideas of variational perturbation theory with rather simple examples from quantum mechanics. The approach is, however, by no means limited to quantum mechanics only. In fact, in the past few years many fascinating applications to field theoretic models and critical phenomena, in particular to the precise calculation of critical exponents, have been worked out by Kleinert and his collaborators. Since these applications are by now far too numerous to be reviewed in this contribution, I refer the reader to several other articles in this chapter of the *Festschrift* dealing with the most successful directions of this ongoing line of research.

Acknowledgments

Before closing this article I wish to thank Hagen Kleinert for his guidance and friendship over many years. Being my "Diplom-" and "Doktorvater" he directed my scientific way at an early stage, and I am very grateful for his advice and suggestions. I remember with a smile often having sit next to him in a seminar or colloquium when, after he had a glance on some notes I gave him earlier, he suddenly passed a small sheet of paper saying just a few words: "Das ist ein Knüller" – "This is sensational". I knew of course that this hardly could be true with all the problems we were trying to solve, but

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perhaps this gives a glimpse on how enthusiastic Hagen Kleinert is about his research work and on how stimulating the atmosphere around him can be. For the years to come I wish him the power to maintain this optimistic attitude and to keep up with – or even further increase – the enormous productivity of the past years reflected at least partially in the many references to his work in this *Festschrift* – Happy 60th Birthday and all the best!

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