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FEYNMAN INTEGRAL ON A GROUP

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If G is a semisimple Lie group and G^C its complexification then we define a stochastic process g_t with values in G^C such that $\langle f(g_t)\rangle$ is a solution of the Schrödinger equation on G. We consider a function V (a potential) continuous on G and meromorphic on G^C. We obtain the Schrödinger evolution in a potential by means of the Feynman formula. We discuss an equivalent construction of the Schrödinger evolution by means of stochastic equations which are defined as a stochastic perturbation of classical dynamics.

1 Introduction

A definition of the Feynman integral on a manifold encounters numerous difficulties. These difficulties arise because there seems to be no natural finite-dimensional approximation to the Feynman integral on a manifold (contrary to the flat case). It should be useful to apply a map of a manifold (locally) into a region in a flat space, where the path integral is well-defined. The method has been applied by mathematicians [1] in order to define the Brownian motion and the Wiener measure on a Riemannian manifold. Multivalued mappings have been introduced by Hagen Kleinert [2]. Such transformations allow us to define the Feynman integral on a manifold with curvature and torsion.

In this contribution we study the mapping method in order to define the Feynman integral on a semisimple Lie group in terms of the Brownian motion on R^d. The Feynman integral is understood here in a broad sense as defining the solution of the Schrödinger equation by means of a functional measure. In our earlier works [3,4] we have studied the Brownian motion

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representation of the Feynman integral in a flat case. The manifolds were only briefly discussed. The semisimple Lie groups are particularly simple from our point of view because there is a natural map from a flat space (the algebra) onto the group. So, we can transform the Brownian motion from the algebra into a complex process on a complexification of the group. This process gives a realization of the Feynman integral. Subsequently, we consider a function V on a group (a potential) and derive the Feynman integral representation of the solution of the Schrödinger equation with the potential. We discuss also the semiclassical expansion and a Langevin equation on a group which can be applied to generate this semiclassical expansion.

2 Notation and Preliminaries

Let G be the d-dimensional Lie group, whose complexification is denoted by G^C. $\mathcal{G}$ is the Lie algebra of G (we denote its complexification by $\mathcal{G}^C$). The exponential map $\mathcal{E}$ is a diffeomorphism of a neighbourhood of 0 in $\mathcal{G}$ into a neighbourhood of 1 in G . It extends to a holomorphic map of a neighbourhood of $\mathcal{G}^C$ into a neighbourhood of G^C. We shall restrict ourselves to matrix groups (which give a faithful representation of a semisimple Lie group G). By τ^j we denote a basis of the Lie algebra $\mathcal{G}$ normalized by the Killing form C^{jk}. Then we define $\Upsilon = \sum_1^d y_j\tau^j$, where $y_j \in R$. When restricted to the matrix group the exponential map will also be denoted by $\mathcal{E}(\Upsilon) = \exp(\Upsilon)$, because in this case the exponential has a direct meaning as an exponential of a matrix (now $\mathrm{Tr}\,(\tau^j\tau^k) = -C^{jk}$).

Let r_j ($j = 1,\ldots,d$), be independent random Gaussian variables. The vector $\mathbf{r}$ can be considered as a random variable in a d-dimensional algebra $\mathcal{G}$ with the probability distribution

$$p(\mathbf{y}) = (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{|\mathbf{y}|^2}{2}\right). \tag{1}$$

We wish to transform this random variable to the group by means of the exponential map. Let $\mathcal{R} = \sum_{k=1}^d r_j\tau^j \in \mathcal{G}$. We define the probability distribution on the semisimple Lie group G^C by the formula

$$E\left[F\left(\exp(z\mathcal{R})\right)\right] = \int_{R^d} d\mathbf{y}p(\mathbf{y})F\left(\exp(z\Upsilon)\right). \tag{2}$$

For the Feynman integral we choose

$$z = \epsilon\lambda\sigma,$$

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where ϵ is a real parameter, λ is defined by

$$\lambda = \frac{1}{\sqrt{2}}(1+i),$$

and

$$\sigma = \sqrt{\frac{\hbar}{m}}$$

with Planck's constant $\hbar$ and mass m.

3 Discretization of the Feynman Integral

The Gaussian random variables are applied in order to define a finite-dimensional approximation to the Feynman integral. We define recursively a random walk on the group as a solution of the equation

$$g(n+1) = \exp(z\mathcal{R}(n))g(n), \tag{3}$$

where $n \geq 0$ and $g(0) = g$. Then, we take the limit $n \to \infty$, $\epsilon \to 0$ and $\epsilon^2 n \to t \geq 0$ in order to define a continuous process $g(t)$ with values in G^C. It has the following properties

i) $g(0) = g$.
ii) The increments

$$g(t_1)^{-1}g(t_2), \ldots, g(t_{n-1})^{-1}g(t_n)\,, \qquad t_1 \leq t_2 \leq ... \leq t_n \tag{4}$$

are mutually independent.
iii) The probability distribution of $g(t_1)^{-1}g(t_2)$ depends only on $t_2 - t_1$.
iv) $g(t)$ is the solution of the Stratonovitch stochastic equation [5]

$$dg(t) = z db(t) \circ g(t), \tag{5}$$

where $b(s) = \sum_{k=1}^{d} \tau^k b_k(s)$ and $b_k(s)$ are independent Brownian motions.

Eq. (5) follows from Eq. (3). In fact, we have

$$\sum_{k=1}^{n} \left(g(k+1) - g(k)\right) g(k)^{-1} = \sum_{k=1}^{n} (\exp(z\mathcal{R}) - 1),$$

and the right-hand side tends to the Brownian motion as $\epsilon \to 0$ and $n \to \infty$ (because the sum of independent random variables tends to the Brownian motion).

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When the probability distribution (2) of independent random variables is known then the distribution of group elements can be formally expressed by a change of variables $y \to g$,

$$\prod_k dy_k \exp(-y_k^2/2) = \prod_k dy_k \exp\left\{-\frac{1}{2z^2}\mathrm{Tr}\left[\ln\left(g\left(k+1\right)g\left(k\right)^{-1}\right)\right]^2\right\}. \quad (6)$$

It is clear that the logarithm is only locally defined around the unit element of the group. Moreover, in order to replace the R^d integral by a group integral, we would need to calculate the Jacobian. Again we could do it only locally. Fortunately, we do not need to derive the corresponding formulas explicitly. We know that in general a linear continuous functional $\langle f(g_1, \ldots, g_n)\rangle$ on a Cartesian product of groups determines a measure on the product of groups. This is the discrete version of the Feynman integral. We discuss still another definition in the next section.

4 Unitary Evolution in $L^2(dg)$

Instead of describing explicitly the discrete Feynman measure we consider the discrete time evolution of the group element. The probability distribution (2) of $g(n)$ determines the operator

$$(K_{\epsilon,\sigma}F)(g) = E\left[F\left(g\left(\mathcal{R}\right)g\right)\right], \quad (7)$$

where we denoted $g(\mathcal{R}) = \exp(\epsilon\lambda\sigma\mathcal{R})$. Clearly,

$$(K^n_{\epsilon,\sigma}F)(g) = E\left[F\left(g\left(\mathcal{R}(n)\right)\ldots g\left(\mathcal{R}(1)\right)g\right)\right], \quad (8)$$

where $\mathcal{R}(k)$ numerates independent random variables. From Eqs. (2), (3), and (7) we obtain the kernel of $K_{\epsilon,\sigma}$

$$(K_{\epsilon,\sigma}F)(g) = \int d\mathbf{y}p(\mathbf{y})F\left(g(\Upsilon)g\right) \equiv \int d\mathbf{y}K_{\epsilon,\sigma}(g,\mathbf{y})F\left(g(\Upsilon)\right). \quad (9)$$

Eq. (7) has a meaning only for analytic functions F. However, using the (distributional) kernel (9) we can extend the definition of $K_{\epsilon,\sigma}$ to any $F \in L^1(dg)$. From Eq. (8) we have

$$(K^n_{\epsilon,\sigma}F)(g) = \int d\mathbf{y}_1\ldots d\mathbf{y}_n p(\mathbf{y}_1)\ldots p(\mathbf{y}_n)F\left(g\left(\Upsilon_n\right)\ldots g\left(\Upsilon_1\right)g\right). \quad (10)$$

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We define the generator A_σ of $K_{\epsilon,\sigma}$ as a limit in $L^2(d\Upsilon)$,

$$A_\sigma = \lim_{\epsilon \to 0} \epsilon^{-2}(K_{\epsilon,\sigma} - 1) \equiv \frac{i\hbar}{2m} \triangle_G. \tag{11}$$

Applying the Taylor formula on the Lie group we can compute the generator A_σ of $K_{\epsilon,\sigma}$ directly from Eq. (11). We obtain

$$A_\sigma = -\frac{i\hbar}{2} \sum_{k=1}^{d} C^{kl} X_k X_l, \tag{12}$$

where X_k is the basis of right-invariant vector fields on G corresponding to the basis $\tau \in \mathcal{G}$. Then $\triangle_G$ is the Laplace-Beltrami operator on the group.

The limit $\epsilon \to 0$ in Eq. (9) defines a continuous time semigroup. In fact, it follows from Eq. (11) that if we define

$$(U_t F)(g') \equiv F_t(g') = E[F(g(t)g')], \tag{13}$$

where $g(t)$ (with $g(0) = 1$) is the solution of Eq. (5), then U_t defines a semigroup in $L^2(dg)$ with the generator A_σ. F_t is the solution of the Schrödinger equation

$$\frac{d}{dt} F_t(g') = A_\sigma F_t(g') \tag{14}$$

with the initial condition F.

After a change of coordinates from the algebra to the group, Eq. (10) could be considered as another rigorous version of the discrete Feynman integral (a formal expression is described by Eq. (6)). Note however that the map $G \to R^d$ is multivalued and discontinuous (as the maps considered in Ref. [6]). Hence, the formula for a change of variables in Eq. (9) would be rather complicated. The Feynman integral on a manifold has been defined in some earlier papers [7–9] in terms of the short-time propagator. However, the short-time propagator must be derived first by other methods. Then, the continuum limit (13) of the products of short-time propagators does not define a measure. Finally, in order to obtain the generator (11) one must artificially subtract the $R/6$ term (where R is the scalar curvature) from the propagator. The $R/6$ term comes from a determinant of the map $R^d \to G$ (as was shown in Ref. [2]). Hence, its presence is explained by the method of non-holonomic equivalence principle [6].

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5 Feynman Integral on a Group

Let V be a continuous function on G which is meromorphic on G^C. We are going to express a solution of the Schrödinger equation with the Hamiltonian $H = i\hbar A_\sigma + V$ by the Feynman integral. Assume that for a function F defined on G^C the function

$$\exp\left(-\frac{i}{\hbar}\int_0^t V\left(g(s)g'\right) ds\right) F(g(t)g')$$

is integrable. Then

$$(U^V(t)F)(g) = E\left[\exp\left(-\frac{i}{\hbar}\int_0^t V\left(g(s)g\right) ds\right) F\left(g(t)g\right)\right] \tag{15}$$

satisfies the Schrödinger equation

$$\frac{d}{dt}F_t(g) = \left(A_\sigma - \frac{i}{\hbar}V(g)\right) F_t(g) \tag{16}$$

with the initial condition $\lim_{t\to 0} F_t(g) = F(g)$.

The semigroup property of the operator defined by the right-hand side of Eq. (15) follows from the Markov property of $g(t)$ and the additivity of $\int_0^t ds\, V$ as a function of t. The Markov property of $g(t)$ follows from its construction (3). Then, using the semigroup property it is sufficient to prove Eq. (16) at $t = 0$. At $t = 0$, Eq. (16) follows from Eq. (14) and the Leibniz rule of differentiation.

There remains the basic problem of showing the integrability of the function inside the square brackets in Eq. (15).

Our basic tool in the proof of integrability is the Jensen inequality

$$\begin{aligned}
&\left|E\left[\exp\left(-\frac{i}{\hbar}\int_0^t V\left(g(s)g\right) ds\right) F\left(g(t)g\right)\right]\right| \\
&\qquad \le \int_0^t \frac{ds}{t} E\left[\exp\left(\Im\frac{t}{\hbar}V\left(g(s)g\right)\right)\left|F\left(g(t)g\right)\right|\right] \\
&\qquad = \int_0^t \frac{ds}{t} dg' K_s(g,g') \exp\left(\Im\frac{t}{\hbar}V\left(g'\right)\right)\left|F\left(g'\right)\right|,
\end{aligned} \tag{17}$$

where K_s is the kernel of U_s. We apply this inequality for a regularized V, e.g.

$$V_R\left(g\left(s\right)\right) = V\left(g\left(s\right)\right)\exp\left(-\int_0^s d\tau \mathrm{Tr}\ \left(b\left(\tau\right)\right)^2/R\right),$$

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which is a bounded function, then we take the limit $R \to \infty$. Hence, the inequality (17) holds true for V itself. Another method of a regularization of the Feynman formula comes from the Trotter product formula (U_t is defined in Eq. (13))

$$U_t^V F = \lim_{n\to\infty} \left[U_{t/n} \exp\left(-\frac{it}{n\hbar} V \right) \right]^n F,$$

where for a large n

$$U_{t/n} F(g) \approx \int d\mathbf{y} p(\mathbf{y}) F\left(\exp\left(\lambda\sigma \frac{t}{n} \Upsilon \right) g \right). \tag{18}$$

Hence

$$\begin{aligned} &\left| U_{t/n} \exp\left[-\frac{it}{n\hbar} V(g) \right] F(g) \right| \\ &\quad \approx \int d\mathbf{y}\, p(\mathbf{y}) \left| \exp\left[\Im \frac{t}{n\hbar} V\left(\exp\left(\lambda\sigma \frac{t}{n} \Upsilon \right) g \right) \right] \right| \\ &\quad\quad \times \left| F\left(\exp\left(\lambda\sigma \frac{t}{n} \Upsilon \right) g \right) \right|. \end{aligned} \tag{19}$$

At this stage we can apply the Jensen inequality (17).

As an example let us consider the group $SU(2)$. Each element of the group can be expressed in the form (where $\mathbf{r} = (y_1, \ldots, y_d)$ and $r = |\mathbf{r}|$)

$$g = \exp(i\mathbf{s}\mathbf{r}) = I \cos r + i\mathbf{s}\mathbf{r} \sin r, \tag{20}$$

where $\mathbf{s}$ are the Pauli matrices satisfying

$$[s_j, s_k] = 2i\epsilon_{jkl} s_l.$$

As a typical meromorphic function on G^C we may consider

$$V(g) = \mathrm{Tr}\,(g)(1 + (\mathrm{Tr}\,(g))^2)^{-1}, \tag{21}$$

where $\mathrm{Tr}\,(g) = 2\cos r$. Now, we have

$$g = \exp(\lambda\sigma\Upsilon) = I \cos(\lambda\sigma r) + i\lambda\sigma\mathbf{s}\mathbf{y} \sin(\lambda\sigma r).$$

In order to prove that the integral (15) is finite for a small time we must show that the singularities of $\exp(\Im V(\exp(\lambda\sigma\Upsilon) g))$ are integrable . Let us take for simplicity $g = 1$ in Eq. (15). Then we have $\mathrm{Tr}\,(\exp(\lambda\sigma\Upsilon)) = 2\cos((1+i)A)$

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for a certain real A. Hence, the denominator in Eq. (15) with the potential (21) is

$$1+(\mathrm{Tr}\,(\exp(\lambda\sigma\Upsilon)))^2 = 1+\cos^2 A\cosh^2 A-\sin^2 A\sinh^2 A \\ -\frac{i}{2}\sin 2A\sinh 2A. \tag{22}$$

We can see that this expression is never zero. Hence, the exponential factor is integrable. In general, the denominator can be zero but on a set of measure zero. Then, the integrability can still be established.

6 Semiclassical Expansion

The classical limit of Eq. (13) (without the potential) is determined by the Lagrangian

$$L=\frac{m}{2}\mathrm{Tr}\left(\frac{dg}{dt}g^{-1}\right)^2. \tag{23}$$

As there is only the kinetic term in L, the time derivative of L vanishes when calculated on the classical solution. Hence, the classical equations read

$$\frac{dg}{dt}g^{-1}=v=\mathrm{const.}$$

We can now construct the action and the semiclassical approximation which appears to be the exact solution of the quantum problem [9]. Hence, without the potential, the classical limit is trivial. It becomes non-trivial if the potential is present.

There are two approaches to the semiclassical expansion:

i) We transform first the Schrödinger equation using a solution of the Hamilton-Jacobi equation, express the solution by a modified Feynman integral and subsequently estimate the remainder.
ii) We make a shift of variables in the Feynman integral using the classical solution and then estimate the remainder.

The methods i)-ii) lead to the same result but they require different assumptions to justify the estimates. Let us begin with the first method. Assume that W_s is the solution (with the initial condition W) of the Hamilton-Jacobi equation on G with a scalar potential V:

$$\partial_s W_s+\frac{1}{2m}(\nabla W_s,\nabla W_s)+V=0. \tag{24}$$

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The scalar product in Eq. (24) is with respect to the Riemannian metric on G. Let us consider a solution of the Schrödinger equation with the initial condition $\psi = \exp(iW/\hbar)\phi$. Then, the solution of the Schrödinger equation can be expressed in the form $\psi_t = \exp(iW_t/\hbar)\phi_t$, where ϕ_t is the solution of the equation

$$\partial_s \phi_s = \frac{i\hbar}{2m} \triangle_G \phi_s - \frac{1}{m}(\nabla W_s, \nabla \phi_s) + \frac{1}{2m}(\triangle_G W_s)\phi_s. \tag{25}$$

On a formal level we may neglect the second-order differential operator in Eq. (25) in the limit $\hbar \to 0$. Then, the limit $\hbar \to 0$ of the solution of Eq. (25) is determined by the classical flow. In order to make the formal argument rigorous as well as to derive an expansion in $\hbar$ it is useful to express the solution of Eq. (25) by a Markov process

$$\phi_t(x) = E\left[\exp\left(\frac{1}{2m}\int_0^t (\triangle_G W_{t-s}(\zeta_s))\, ds\right) \phi\left(\zeta_t(x)\right)\right], \tag{26}$$

where the process $\zeta_s(x)$ is defined for $0 \leq s \leq t$ as a solution of the Langevin equation (with the initial condition $x \in G$)

$$d\zeta^\mu(s) = -\frac{1}{m} g^{\mu\nu}(\zeta(s))\partial_\nu W_{t-s}(\zeta(s))ds + d\xi^\mu(s), \tag{27}$$

where $g^{\mu\nu}$ is the Riemannian metric on G and ξ denotes the Markov process generated by $i\hbar\triangle_G/2m$ defined in a coordinate independent way in Eq. (5). In the classical limit $\xi \to 0$ the solution of Eq. (27) converges to the classical trajectory of a particle moving on a group manifold in the potential V.

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References

[1] D.K. Elworthy and A. Truman, *J. Math. Phys.* **22**, 2144 (1981).

Fluctuating Paths and Fields, Eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (World Scientific, Singapore, 2001).

[2] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995).
[3] Z. Haba, *J. Phys. A* **27**, 6457 (1994).
[4] Z. Haba, *Feynman Integral and Random Dynamics in Quantum Physics* (Kluwer, Dordrecht, 1999).
[5] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North-Holland, Amsterdam, 1981).
[6] H. Kleinert, *Gen. Rel. Grav.* **32**, 769 (2000).
[7] B.S. DeWitt, *Rev. Mod. Phys.* **29**, 377 (1957).
[8] D.W. McLaughlin and L.S. Schulman, *J. Math. Phys.* **12**, 2520 (1971).
[9] J.S. Dowker, *Ann. Phys. (N.Y.)* **62**, 361 (1971).