
ANALYTIC CONFINEMENT AND REGGE SPECTRUM

G.V. EFIMOV

*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Dubna, Russia*

E-mail:efimovg@thsun1.jinr.ru

Using a simple relativistic QFT model of scalar fields, we demonstrate that the analytic confinement, where propagators of initial constituent particles are entire analytic functions in the complex p^2 -plane, and the weak coupling constant lead to the linear Regge behavior of two- and three-particle bound states.

1 Introduction

The problem of Regge trajectories in particle physics is an active area of investigation beginning in the 1960's [1]. Experimental data show that meson and baryon Regge trajectories are almost linear, though the latter can only be approximatively linear [2]. Standard theoretical calculations which give the linear spectrum for meson and baryon mass squares are based on a relativized Schrödinger equation with an appropriate linearly increasing potential [3,4]. Perturbative QCD approaches have shown that Regge trajectories are non-linear [5,6].

We show in this article, using a simple relativistic model of scalar fields, that in the framework of QFT analytic confinement of constituent particles (propagators are entire analytic functions in the momentum p^2 -space) and weak coupling constant (the Bethe-Salpeter equation can be used) lead to the linear Regge spectrum of bound states.

Thus, if the QCD vacuum results in analytic confinement of quarks and gluons [7–9] and the QCD coupling constant α_s is small, hadron bound states are expected to have, at least asymptotically, the linear Regge spectrum, and masses of these states can be calculated by the Bethe-Salpeter equation.

2 The Model

We consider a simple quantum field model with confinement and demonstrate the properties of possible bound states that can be interpreted as standard physical particles. Let $\Phi(x)$ and $\varphi(x)$ be two scalar fields with the Lagrangian in the Euclidean metrics

$$\begin{aligned} \mathcal{L}(x) = & -\Phi^+(x)S^{-1}(\square)\Phi(x) - \frac{1}{2}\varphi(x)D^{-1}(\square)\varphi(x) \\ & - g\Phi^+(x)\Phi(x)\varphi(x) - g\varphi^3(x) \end{aligned} \quad (1)$$

with

$$S^{-1}(\square) = \frac{\Lambda^2}{\epsilon} \cdot e^{\frac{\square}{\Lambda^2}}, \quad D^{-1}(\square) = \Lambda^2 e^{\frac{\square}{\Lambda^2}}.$$

The equation for a free field $A(x) = (\Phi(x), \varphi(x))$ looks like

$$e^{\frac{\square}{\Lambda^2}}A(x) = 0, \quad \text{or} \quad e^{-\frac{p^2}{\Lambda^2}}\tilde{A}(p) = 0. \quad (2)$$

The solutions are $A(x) \equiv 0$ ($\varphi(x) \equiv 0$ and $\Phi(x) \equiv 0$), because the function $e^{-p^2/\Lambda^2} \neq 0$, i.e. it has no zeroes for any real or complex p^2 . Exactly this property means analytic confinement. The scale of the confinement region is characterized by the parameter Λ . The fields $\Phi(x)$ and $\varphi(x)$ can exist in a virtual state only, so that they can be called *virton fields* [11]. In addition, one can say that the field $\Phi(x)$ describes constituent particles (scalar “quarks”), and the field $\varphi(x)$ particles-carriers (scalar “gluons”). The parameter $\epsilon \ll 1$ means that “the mass” of “the quark” Φ is much larger than “the mass” of “the gluon” ϕ . In other words, the Compton wavelength of “the massive quark” Φ is smaller than the diameter of the confinement region.

The propagators in momentum and x -spaces look like

$$\begin{aligned} \tilde{S}(p^2) = & \frac{\epsilon}{\Lambda^2} e^{-\frac{p^2}{\Lambda^2}}, \quad \tilde{D}(p^2) = \frac{1}{\Lambda^2} e^{-\frac{p^2}{\Lambda^2}}, \\ S(x) = & \frac{\Lambda^2\epsilon}{(4\pi)^2} e^{-\frac{\Lambda^2}{4}x^2}, \quad D(x) = \frac{\Lambda^2}{(4\pi)^2} e^{-\frac{\Lambda^2}{4}x^2}. \end{aligned} \quad (3)$$

They are entire analytic functions in the complex p^2 -plane. This guarantees the confinement of “particles” $\Phi(x)$ and $\varphi(x)$ in each perturbation order in the dimensionless coupling constant

$$\lambda = \left(\frac{g}{4\pi\Lambda} \right)^2 \leq 1. \quad (4)$$

The mechanism of bound states can be described in the following way [10]. Let us consider the partition function

$$Z = \iiint \delta\Phi\delta\Phi^+\delta\varphi e^{-(\Phi^+S^{-1}\Phi) - \frac{1}{2}(\varphi D^{-1}\varphi) - g(\Phi^+\Phi\varphi) - g\varphi^3}.$$

After integration over ϕ in the one-particle exchange approximation, valid for small coupling constants λ , Z reads

$$Z = \iint \delta\Phi\delta\Phi^+ e^{-(\Phi^+S^{-1}\Phi) + L_2[\Phi] + L_3[\Phi] + O[\Phi^8]}, \quad (5)$$

where

$$\begin{aligned} L_2[\Phi] &= g^2(\Phi_1^+\Phi_1D_{12}\Phi_2^+\Phi_2), \\ L_3[\Phi] &= g^4(\Phi_1^+\Phi_1\Phi_2^+\Phi_2\Phi_3^+\Phi_3\Gamma_{123}). \end{aligned}$$

Here,

$$\Phi_j^+ = \Phi(x_j), \quad D_{ij} = D(x_i - x_j),$$

and

$$\Gamma_{123} = \Gamma(x_1, x_2, x_3) = \int dy D(x_1 - y)D(x_2 - y)D(x_3 - y),$$

where the integration over dx_j ($j = 1, 2, 3$) is implied.

The term

$$L_4[\Phi] = g^4(\Phi_1^+\Phi_1D_{12}\Phi_2^+S_{23}\Phi_3D_{34}\Phi_4^+\Phi_4)$$

is important in the Faddeev equations for a three-body problem. In our case, it is small in comparison with L_3 according to our assumption $\epsilon \ll 1$.

3 Two-Particle Bound States

Two-particle bound states are defined by the term L_2 which can be transformed as

$$L_2 = g^2(\Phi_1^+\Phi_2\sqrt{D_{12}} \cdot \sqrt{D_{12}}\Phi_2^+\Phi_1).$$

Let us use the Gaussian representation

$$e^{L_2} = \iint \delta B\delta B^+ e^{-(B_{12}^+B_{12}) + g[(B_{12}^+\Phi_1^+\Phi_2\sqrt{D_{12}}) + (\sqrt{D_{12}}\Phi_2^+\Phi_1B_{12})]},$$

where the bilocal field $B_{12} = B(x_1, x_2)$ is introduced.

We substitute this representation into the partition function (5) and integrate over Φ . The result is

$$Z = \iint \delta B \delta B^+ e^{-(B_{12}^+ B_{12}) + 2g^2 (B_{12} \sqrt{D_{12}} S_{11'} S_{22'} \sqrt{D_{1'2'}} B_{1'2'}) + O[g^3]} \quad (6)$$

with $S_{11'} = S(x_1 - x'_1)$, and so on. We introduce the variables

$$x_1 = x + \xi, \quad x_2 = x - \xi$$

and denote $B(x_1, x_2) = \frac{1}{4} \mathcal{B}(x, \xi)$. After some calculations with (3) we get

$$\begin{aligned} & 2g^2 (B_{12} \sqrt{D_{12}} S_{11'} S_{22'} \sqrt{D_{1'2'}} B_{1'2'}) \\ &= \frac{32g^2 \Lambda^4 \epsilon^2}{(4\pi)^4} \iint dx dx' \iint d\xi d\xi' \mathcal{B}(x, \xi) e^{-\frac{\Lambda^2}{2}(x-x')^2} K(\xi, \xi') \mathcal{B}(x', \xi'). \end{aligned}$$

Here the kernel $K(\xi, \xi')$ can be diagonalized

$$\begin{aligned} K(\xi, \xi') &= e^{-\Lambda^2(\xi^2 - (\xi\xi') + \xi'^2)} = \sum_Q U_Q(\xi) \kappa_Q U_Q(\xi'), \quad (7) \\ \kappa_Q &= \kappa_{nl} = \left(\frac{2\pi}{\Lambda^2(2 + \sqrt{3})} \right)^2 \cdot \left(\frac{1}{2 + \sqrt{3}} \right)^{2n+l} \end{aligned}$$

(for the eigenfunctions $U_Q(\xi)$ see the Appendix).

The numbers $Q = (nl\{\mu\})$ can be considered as radial n , orbital l , and magnetic $\{\mu\} = (\mu_1, \dots, \mu_l)$ quantum numbers.

It should be stressed that the diagonalization of the kernel $K(\xi, \xi')$ is nothing else but the solution of the Bethe-Salpeter equation in the one-boson exchange when the propagators are defined by (3). To go to the standard form of the Bethe-Salpeter equation, we have to introduce a new function $U_Q(y) = \sqrt{D(y)} \Psi_Q(y)$ and to pass to the momentum representation [10].

Let us introduce the function

$$\mathcal{B}_Q(x) = 4 \int d\xi U_Q(\xi) \mathcal{B}(x, \xi), \quad \tilde{\mathcal{B}}_Q(p) = \int dx e^{ipx} \mathcal{B}_Q(x). \quad (8)$$

Then the Gaussian quadratic measure defining the partition function (6) gets the form

$$Z = \prod_Q \delta \tilde{\mathcal{B}}_Q \tilde{\mathcal{B}}_Q^+ \exp \left\{ - \int \frac{dp}{(2\pi)^4} \sum_Q \tilde{\mathcal{B}}_Q^+(p) [1 - \Pi_Q(p^2)] \tilde{\mathcal{B}}_Q(p) \right\} \quad (9)$$

with

$$\Pi_Q(p^2) = \frac{\lambda}{\lambda_{2c}} \left(\frac{1}{2 + \sqrt{3}} \right)^{2n+l} e^{-\frac{p^2}{2\Lambda^2}}, \quad \lambda_{2c} = \frac{(2 + \sqrt{3})^2}{2\epsilon^2}. \quad (10)$$

The equation

$$1 = \Pi_Q(-M_Q^2) \quad (11)$$

defines the mass $M_Q = M_{nl}$ of two-particle bound states with quantum numbers Q :

$$M_Q^2 = M_{nl}^2 = \Lambda^2 2 \ln \frac{\lambda_{2c}}{\lambda} + (2n + l)\Lambda^2 2 \ln(2 + \sqrt{3}). \quad (12)$$

One can see that this spectrum has a purely linear Regge behavior.

The function $1 - \Pi_Q(p^2)$ defines the Gaussian measure or kinetic term in the functional integral (9). To represent this function in the standard form, let us develop it in the vicinity of the point $p^2 = -M_Q^2$:

$$1 - \Pi_Q(p^2) = Z_Q(p^2 + M_Q^2) + O((p^2 + M_Q^2)^2)$$

with

$$Z_Q = -\Pi'_Q(-M_Q^2) = \frac{\lambda}{\lambda_c} \cdot \frac{e^{\frac{M_Q^2}{2\Lambda^2}}}{(2 + \sqrt{3})^{l+2n}} \cdot \frac{1}{2\Lambda^2}.$$

Thus, the kinetic term in Z [Eq. (9)] reads

$$\begin{aligned} & \left(\tilde{\mathcal{B}}_Q^+(p) [1 - \Pi_Q(p^2)] \tilde{\mathcal{B}}_Q(p) \right) \\ & = Z_Q \left(\tilde{\mathcal{B}}_Q^+(p) [(p^2 + M_Q^2) + O((p^2 + M_Q^2)^2)] \tilde{\mathcal{B}}_Q(p) \right), \end{aligned}$$

and after the renormalization

$$\tilde{\mathcal{B}}_Q(p) \rightarrow Z_Q^{-1/2} \tilde{\mathcal{B}}_Q(p)$$

it gets the standard form.

4 Three-Particle Bound States

Three-particle bound states are defined by the term $L_3[\Phi]$ in (5) which can be transformed as

$$L_3 = g^4 (\Phi_1^+ \Phi_2^+ \Phi_3^+ \sqrt{\Gamma_{123}} \cdot \sqrt{\Gamma_{123}} \Phi_1 \Phi_2 \Phi_3)$$

with

$$\Gamma_{123} = \Gamma(x_1, x_2, x_3) = \frac{\Lambda^2}{9(4\pi)^4} e^{-\frac{\lambda^2}{2}(3\xi_1^2 + \xi_2^2)},$$

where

$$x_1 = x + \xi_1 + \xi_2, \quad x_2 = x + \xi_1 - \xi_2, \quad x_3 = x - 2\xi_1. \quad (13)$$

We use the Gaussian representation

$$e^{L_3} = \iint \delta H \delta H^+ e^{-(H_{123}^+ H_{123}) + g^4 [(H_{123}^+ \sqrt{\Gamma_{123}} \Phi_1 \Phi_2 \Phi_3) + (\Phi_1^+ \Phi_2^+ \Phi_3^+ \sqrt{\Gamma_{123}} H_{123})]},$$

where $\Phi_j = \Phi(x_j)$, $j = 1, 2, 3$, and $H = H_{123} = H(x_1, x_2, x_3)$ are tri-local fields. We substitute this representation into (5). After integration over Φ , the partition function reads

$$Z = \iint \delta \Phi \delta \Phi^+ e^{-(\Phi^+ S^{-1} \Phi) + L_3} = \iint \delta H \delta H^+ e^{-(H_{123}^+ H_{123}) + g^4 W[H] + O(g^6)},$$

where

$$g^4 W[H] = 6g^4 (H_{123}^+ \sqrt{\Gamma_{123}} S_{11'} S_{22'} S_{33'} \sqrt{\Gamma_{1'2'3'}} H_{1'2'3'}).$$

Using the notation

$$\tilde{H}(p; \xi) = \tilde{H}(p; \xi_1, \xi_2) = \int dx e^{ipx} H(x + \xi_1 + \xi_2, x + \xi_1 - \xi_2, x - 2\xi_1),$$

one can get after some calculations

$$g^4 W[H] = \lambda^2 C \int \frac{dp}{(2\pi)^4} \iint d\xi \iint d\xi' \tilde{H}^+(p; \xi) \Sigma(p; \xi, \xi') \tilde{H}(p; \xi'),$$

$$\Sigma(p; \xi, \xi') = e^{-\frac{p^2}{3\lambda^2}} \cdot K_1(\xi_1, \xi'_1) \cdot K_2(\xi_2, \xi'_2),$$

where C is a constant and

$$K_j(\xi_j, \xi'_j) = e^{-\frac{c_j \lambda^2}{4} [3\xi_j^2 - 4\xi_j \xi'_j + \xi_j'^2]} = \sum_{Q_j} U_{Q_j}^{(j)}(\xi_j) \kappa_{Q_j}^{(j)} U_{Q_j}^{(j)}(\xi'_j), \quad (j = 1, 2),$$

with $c_1 = 3$ and $c_2 = 1$, and

$$\kappa_{Q_j}^{(j)} = \left(\frac{4\pi}{c_j \Lambda^2 (3 + \sqrt{5})} \right)^2 \left(\frac{2}{3 + \sqrt{5}} \right)^{2n_j + l_j}.$$

Thus, we get

$$(H_{123}^+ H_{123}) = \int \frac{dp}{(2\pi)^4} \sum_{Q_1 Q_2} \tilde{H}_{Q_1 Q_2}^+(p) \tilde{H}_{Q_1 Q_2}(p),$$

$$g^4 W[H] = \int \frac{dp}{(2\pi)^4} \sum_{Q_1 Q_2} \tilde{H}_{Q_1 Q_2}^+(p) \Sigma_{Q_1+Q_2}(p^2) \tilde{H}_{Q_1 Q_2}(p),$$

where

$$\tilde{H}_{Q_1 Q_2}(p) = \iint d\xi_1 d\xi_2 U_{Q_1}^{(1)}(\xi_1) U_{Q_2}^{(2)}(\xi_2) \tilde{H}(p; \xi_1, \xi_2),$$

$$\Sigma_{Q_1+Q_2}(p^2) = \Sigma_{nl} = e^{-\frac{p^2}{3\Lambda^2}} \cdot \left(\frac{\lambda}{\lambda_{3c}} \right)^2 \cdot \left(\frac{2}{3 + \sqrt{5}} \right)^{2(n_1+n_2)+(l_1+l_2)},$$

$$\lambda_{3c}^2 = \frac{3(3 + \sqrt{5})^4}{2^5 \epsilon^3}, \quad n = n_1 + n_2, \quad l = l_1 + l_2.$$

The Gaussian quadratic measure over the fields $\tilde{\mathcal{H}}_{Q_1 Q_2}(p) = \tilde{\mathcal{H}}_Q(p)$ looks like

$$\prod_Q \delta \tilde{\mathcal{H}}_Q \tilde{\mathcal{H}}_Q^+ \exp \left\{ - \int \frac{dp}{(2\pi)^4} \sum_Q \tilde{\mathcal{H}}_Q^+(p) [1 - \Sigma_{nl}(p^2)] \tilde{\mathcal{H}}_Q(p) \right\}. \quad (14)$$

Here $Q = \{Q_1, Q_2\}$ is the condensed index and $n = n_1 + n_2$ and $l = l_1 + l_2$.

The equation

$$1 = \Sigma_{nl}(-\mathcal{M}_{nl}^2) \quad (15)$$

defines the mass $\mathcal{M}_Q = \mathcal{M}_{nl}$ of three-particle bound states with quantum numbers $Q = nl\{\mu\}$. The spectrum has a purely linear Regge behavior:

$$\mathcal{M}_{nl}^2 = \Lambda^2 3 \ln \frac{\lambda_{3c}}{\lambda} + (2n + l) \Lambda^2 3 \ln \left(\frac{3 + \sqrt{5}}{2} \right). \quad (16)$$

We observe that the states $\tilde{\mathcal{H}}_Q(p) = \tilde{\mathcal{H}}_{Q_1 Q_2}(p)$ are degenerated, so that the states with fixed sums $n = n_1 + n_2$ and $l = l_1 + l_2$ have the same mass.

Now we can represent

$$1 - \Sigma_{nl}(p^2) = Z_{nl}(p^2 + \mathcal{M}_Q^2) + O((p^2 + \mathcal{M}_Q^2)^2),$$

and after the renormalization,

$$\tilde{\mathcal{H}}_Q(p) \rightarrow Z_Q^{-1/2} \tilde{\mathcal{H}}_Q(p)$$

has the standard form.

5 Conclusion

Finally, we can conclude that the analytic confinement leads to a reasonable picture of bound states. Bound states exist for small coupling constants $\lambda < \lambda_c < 1$, and their masses grow when $\lambda \rightarrow 0$ as

$$M_Q \sim \Lambda \sqrt{\ln \frac{\lambda_c}{\lambda}}.$$

This means that the size of the confinement region $r_{\text{cnf}} \sim 1/\Lambda$ exceeds remarkably the Compton length of all bound states for small λ . In other words, the physical particles described by the fields $B_Q(x)$ and $H_Q(x)$ and all physical transformations with them take place in the confinement region.

Analytic confinement leads to the purely linear Regge spectrum for all two- and three-particle bound states with quantum numbers $Q = (nl)$, and the slope of the Regge trajectories is defined by the scale of the confinement region Λ and does not depend on the coupling constant λ . Besides, the slopes of two- and three-particle Regge trajectories are close to each other

$$M_{nl}^2 \sim (2n+l) \cdot \Lambda^2 \cdot 2.633\dots, \quad \mathcal{M}_{nl}^2 \sim (2n+l) \cdot \Lambda^2 \cdot 2.887\dots$$

This idea can be used in QCD. Let us assume that gluon vacuum QCD is realized by the selfdual homogeneous classical field. This assumption leads to analytic confinement of quarks and gluons, and we get the linear Regge spectrum for quark and glueball bound states [7,8,12].

Appendix

Let us consider the kernel

$$K = K(y_1, y_2) = e^{-ay_1^2 + 2by_1y_2 - ay_2^2}, \quad a > b, \quad (17)$$

with

$$\text{Tr } K = \int dy K(y, y) = \int dy e^{-2(a-b)y^2} = \frac{\pi^2}{4(a-b)^2} < \infty.$$

The eigenfunctions of the problem

$$KU_Q = \kappa_Q U_Q, \quad Q = \{nl\{\mu\}\} = \{nl\{\mu_1 \dots \mu_l\}\}$$

or

$$\int dy_2 K(y_1, y_2) U_Q(y_2) = \kappa_Q U_Q(y_1)$$

are the following

$$U_Q = U_{nl\{\mu\}}(y) = N_{nl} T_{l\{\mu\}}(n_y) (y^2)^{\frac{l}{2}} L_n^{(l+1)}(2\beta y^2) e^{-\beta y^2}. \quad (18)$$

Here, $n_y = y/\sqrt{y^2}$ and

$$\beta = \sqrt{a^2 - b^2}, \quad N_{nl} = \frac{\sqrt{2^l(l+1)}}{\pi} (2\beta)^{1+\frac{l}{2}} \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+l+2)}}.$$

The functions $T_{l\{\mu\}}(n)$ satisfy the conditions

$$T_{l\{\mu_1 \mu_2 \dots \mu_l\}}(n) = T_{l\{\mu_2 \mu_1 \dots \mu_l\}}(n), \quad T_{l\{\mu \mu \mu_3 \dots \mu_l\}}(n) = 0,$$

$$\sum_{\{\mu\}} T_{l\{\mu\}}(n_1) T_{l\{\mu\}}(n_2) = \frac{1}{2^l} C_l^1((n_1 n_2)), \quad C_l^1(1) = l + 1,$$

where $C_l^1(t)$ are the Gegenbauer polynomials, and

$$\int dn T_{l\{\mu\}}(n) T_{l\{\mu'\}}(n) = \delta_{ll'} \delta_{\{\mu\}\{\mu'\}} \frac{2\pi^2}{2^l(l+1)}.$$

The eigenvalues are

$$\kappa_Q = \kappa_{nl} = \kappa_0 \cdot \left(\frac{b}{a + \sqrt{a^2 - b^2}} \right)^{2n+l}, \quad \kappa_0 = \frac{\pi^2}{(a + \sqrt{a^2 - b^2})^2}. \quad (19)$$

References

- [1] P.D.B. Collins and E.J. Squires, *Regge Poles in Particle Physics* (Springer-Verlag, Berlin, 1968).
- [2] A. Tang and J.W. Norbury, *Phys. Rev. D* **62**, 016006 (2000).
- [3] S. Godfrey and N. Isgur, *Phys. Rev. D* **32**, 189 (1985).
- [4] D. Kahana, K. Maung Maung, and J.W. Norbury, *Phys. Rev. D* **48**, 3408 (1993).
- [5] E. Di Salvo, L. Kondratyuk, and P. Saracco, *Z. Phys. C* **69**, 149 (1995).

- [6] W.K. Tang, *Phys. Rev. D* **48**, 2019 (1993).
- [7] G.V. Efimov and S.N. Nedelko, *Phys. Rev. D* **51**, 174 (1995); *Eur. Phys. J. C* **1**, 343 (1998).
- [8] Ja.V. Burdanov, G.V. Efimov, S.N. Nedelko, and S.A. Solunin, *Phys. Rev. D* **54**, 4483 (1996).
- [9] G.V. Efimov, A.C. Kalloniatis, and S.N. Nedelko, *Phys. Rev. D* **59**, 014026 (1999).
- [10] G.V. Efimov, eprint: hep-ph/9907483.
- [11] G.V. Efimov and M.A. Ivanov, *The Quark Confinement Model of Hadrons* (IOP Publishing, London, 1993).
- [12] Ja.V. Burdanov and G.V. Efimov, eprint: hep-ph/0009027.