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THE USE OF NON-RELATIVISTIC PATH INTEGRALS IN FIELD THEORIES

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We list a few examples on the application of non-relativistic path integrals to field theoretical problems.

In 1979, I had the fortune to collaborate with Hagen Kleinert. It was the time, when we solved the path integral for the hydrogen atom [1]. Before that, only exact solutions of the path integrals for the free particle and systems with quadratic [2] and inverse quadratic [3] potentials were known. After we found the procedure to calculate the path integral for the H-atom [1], however, almost all of the exactly solvable quantum mechanics text book problems were also solved by path integrals [4]. The basic methods are derived from the H-atom solution: Point canonical transformations, time parameter changes and introduction of auxiliary dynamics in additional space dimensions, reflections from the boundaries. Only the employment of the underlying symmetry group spaces [5] does not have its roots in the techniques used in H-atom path integral.

Solving the quantum mechanical problems by path integrals which have already been solved by the conventional Schrödinger method has its own interest. However apart from its obvious beauty, the path-integration method is suitable for discovering the connections between seemingly different potentials in the Schrödinger picture. It also gives the wave function normalization quite automatically.

The purpose of this note is to mention the use of the exact path-integral solutions for the non-relativistic quantum mechanical point particle motions in some relativistic field theoretical problems.

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Particle pair creations in given classical environments are good examples. As a specific case let us consider the Robertson-Walker cosmologies with the metric

$$ds^2 = -dt^2 + a^2(t) dx^2, \tag{1}$$

where the expansion factor $a(t)$ is a given function of time.

The amplitude A_{ij} for a particle created by the gravitational field at time t in states $f_i(x)$ and $f_j(x)$ is given by

$$\begin{aligned} A_{ij} = -A_0 \int d^3k_a d^3k_b \int d\sigma^\mu(x_a) d\sigma^\nu(x_b) \\ \times f_i^*(\vec{k}_a, x_a) \overleftrightarrow{\partial}^a_\mu f_j^*(\vec{k}_b, x_b) \overleftrightarrow{\partial}^b_\nu K(x_b, x_a). \end{aligned} \tag{2}$$

Here A_0 is the no-particle production amplitude, and $d\sigma^\mu$ is the element of the constant t hypersurface. $K(x_b, x_a)$ is the propagator for a scalar particle of mass μ to go from the space-time point x_b to x_a.

Developing the propagator $K(x_b, x_a)$ by path integrals is important, for it avoids the determination of the initial particle states which is problematic because of the cosmological singularity [6].

The propagator $K(x_b, x_a)$ is given by [1]

$$K(x_b, x_a) = \int_0^\infty dW e^{-i\mu^2 W} F(W, x_b, x_a), \tag{3}$$

where W is the total parameter time. $F(W, x_b, x_a)$ is the Green function which can be expressed as a Hamiltonian path integral as

$$F(W, x_b, x_a) = \int \mathcal{D}^4 x \mathcal{D}^4 k \exp\left\{ i \int_0^W d\omega \left[k_t \dot{t} + \vec{k} \cdot \dot{\vec{x}} + k_t^2 - \frac{\vec{k}^2}{a^2(t)} \right] \right\}, \tag{4}$$

which is formally the same as the non-relativistic particle motion taking place in "space" $(t, \vec{x})$ in the "time" ω. After performing the trivial space integrations over $\mathcal{D}^3 x \mathcal{D}^3 k$ one arrives at

$$\begin{aligned} F(W, x_b, x_a) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}_a - \vec{x}_b)} \int \mathcal{D}t \mathcal{D}k_t \\ \times \exp\left\{ i \int_0^W d\omega \left[k_t \dot{t} + k_t^2 - \frac{\vec{k}^2}{a^2(t)} \right] \right\}. \end{aligned} \tag{5}$$

The path integral in the above expression is equivalent to the one written for the "mass"$= -1/2$, taking place in the (1+1)-dimensional "space-time" (t, ω),

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under the influence of the potential $\vec{k}^2/a^2(t)$, where $\vec{k}^2$ is constant through the path integration. Thus for example for the inflationary Universe with $a(t) = e^{-Ht}$, the problem at hand is equivalent to the exponential potential whose exact solution is known from the Morse potential. For the radiation-dominated Universe with $a(t) \sim \sqrt{t}$ on the other hand, the potential is of the one-dimensional Kepler type that is again exactly solvable.

The second type of field theoretical example I would like to mention briefly is the decay of the vacuum in an external magnetic field. In this case the propagator one needs is expressible as the following type of phase-space path integral

$$G(x_b, x_a) = \int_0^\infty dW e^{-i\mu^2 W} \int \mathcal{D}^4 x \mathcal{D}^4 k \exp\left[i \int_0^W d\omega (k_t \dot{t} + \vec{k} \cdot \dot{\vec{x}} + \vec{k}_t^2 - \vec{k}^2 + eA^\nu \dot{x}_\nu) \right], \qquad (6)$$

with A^ν being the external electromagnetic potential. The path integral in the above Green function is again formally the same as the one written for a non-relativistic particle motion taking place in $(3+1)$-dimensional "space" and in "time" ω.

For specific cases one gets even simpler forms: For example for constant electric field $\vec{E} = E\hat{x}$ in time dependent gauge, i.e. with $A_\mu = (0, Et, 0, 0)$, one ends up formally with a $(1+1)$-dimensional quadratic potential [7]. For the magnetic monopole field the path integral one has in hand is of Pöschl-Teller type in the angular coordinates and the inverse square potential in radial coordinates [8].

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