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CHARACTERIZING VOLUME FORMS

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We present old and new results for characterizing volume forms. The Cartier/DeWitt-Morette group, which regularly shares questions and findings, hopes that Hagen Kleinert will enjoy a new approach for characterizing volume forms on Riemannian and symplectic manifolds, using integration by parts.

1 The Wiener measure

Defining volume forms on infinite-dimensional spaces is a key problem in the theory of functional integration. The first volume form used in functional integration has been the Wiener measure. From the several equivalent definitions of the Wiener measure, we choose one [1] which can easily be extended for use in Feynman integrals. We recall the Cameron-Martin and Malliavin formulae because they are, respectively, integrated and infinitesimal formulae for changes of variable of integration which can be imposed on volume forms other than the Wiener measure.

1.1 Definition

The Wiener measure γ on the space $\mathbb{W}$ of pointed continuous paths w on the time interval $T = [0, 1]$,

$$w : [0,1] \longrightarrow \mathbb{R}\,, \qquad w(0) = 0,$$

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can be characterized by the equation

$$\int_{\mathbb{W}} d\gamma(w)\ \exp\left(-i\langle w', w\rangle\right) = \exp\left(-\frac{1}{2}\int_T\int_T dw'(t)\ dw'(t')\ \inf(t,t')\right), \quad (1)$$

where w' is an element of the topological dual $\mathbb{W}'$ of $\mathbb{W}$, i.e. a bounded measure on the semi-open interval $T =]0, 1]$,

$$\langle w', w\rangle = \int_T dw'(t)\ w(t)\ . \quad (2)$$

1.2 Cameron-Martin Formula

The Cameron-Martin formula can be written as

$$\int_{\mathbb{W}} d\gamma(w)\ F(w+\varphi) = \int_{\mathbb{W}} d\gamma(w)\ J(\varphi, w)\ \ F(w), \quad (3)$$

where $J(\varphi, w)$ is the Radon-Nikodym derivative

$$J(\varphi, w) = \frac{d\gamma(w-\varphi)}{d\gamma(w)}\ . \quad (4)$$

This formally obvious expression for $J(\varphi, w)$ has a not-so-obvious explicit expression. When $\varphi \in L^{2,1}$, i.e. when $\dot\varphi(t)$ is square integrable, and $\varphi(0) = 0$, then[a]

$$J(\varphi, w) = \exp\left(-\frac{1}{2}\int_T dt\ \dot\varphi(t)^2 + \int_T dw(t)\ \dot\varphi(t)\right). \quad (5)$$

The meaning of the second term is subtle since w is not of bounded variation. If $\varphi \in C^2(0,1)$ with the boundary conditions $\varphi(0) = \dot\varphi(1) = 0$, the second term can be integrated by parts, giving $-\int_T dt\ \ w(t)\,\ddot\varphi(t)$, and can then be extended by continuity in the $L^{2,1}$ norm for φ.

A heuristic proof of the Cameron-Martin formula makes the explicit expression (5) "obvious". Let us write formally

$$d\gamma(w) = \mathcal{D}w\ \ \exp\left(-\pi Q(w)\right), \quad (6)$$

[a]The boundary condition $\varphi(0) = 0$ is required in order that $w + \varphi$ belongs to $\mathbb{W}$ when w does, since $w(0) = 0$ for every element w of $\mathbb{W}$.

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where $\mathcal{D}$ is a translation-invariant symbol[b]

$$\mathcal{D}(w+\varphi) = \mathcal{D}w \tag{7}$$

and where Q follows from the definition (1) of the Wiener measure:

$$\int_{\mathbb{W}} \mathcal{D}w \ \exp\left(-\pi Q(w)\right) \ \exp\left(-2\pi i \langle w', w\rangle\right) = \exp\left(-\pi W(w')\right) \tag{8}$$

with

$$W(w') = 2\pi \int_T \int_T dw'(t) \ dw'(t') \ \inf(t,t') \ . \tag{9}$$

By analogy with the finite-dimensional case (43), the quadratic form Q on $\mathbb{W}$ is required to be the inverse of W on $\mathbb{W}'$ in the following sense. Represent W and Q as

$$W(w') = \langle w', Gw'\rangle \qquad \text{and} \qquad Q(w) = \langle Dw, w\rangle; \tag{10}$$

then Q is said to be the inverse of W if

$$DG = \mathbb{1} \ . \tag{11}$$

It follows from (9) and (10) that

$$Gw'(t) = 2\pi \int_T dw'(t') \ \inf(t,t') \tag{12}$$

and from (10) and (11) that

$$Q(w) = \frac{1}{2\pi} \int_T dt \ \left(\frac{dw(t)}{dt}\right)^2 = \frac{1}{2\pi} \int_T \frac{\left(dw(t)\right)^2}{dt} \ . \tag{13}$$

The Cameron-Martin formula (3) is now the obvious statement

$$\int_{\mathbb{W}} \mathcal{D}w \ \exp\left[-\pi Q(w)\right] \ F(w+\varphi) = \int_{\mathbb{W}} \mathcal{D}w \ \exp\left[-\pi Q(w-\varphi)\right] \ F(w), \tag{14}$$

that is $J(\varphi, w) = \exp\left\{\pi\left[Q(w) - Q(w-\varphi)\right]\right\}$. We calculate

$$Q(w-\varphi) = \frac{1}{2\pi} \int_T dt \ \left(\frac{d}{dt}(w-\varphi)(t)\right)^2$$

[b] The symbol $\mathcal{D}$ is often used in physics where $\mathcal{D}w := \prod_t dw(t)$. Here it is defined by (8), which in finite dimensions reduces to (45).

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$$= \frac{1}{2\pi}\left(Q(w) + \int_T dt\ \dot\varphi(t)^2 - 2\int_T dw(t)\ \dot\varphi(t)\right) . \qquad (15)$$

Thus formula (5) follows immediately and this completes the heuristic demonstration.

1.3 An Analogy with the Dobrushin-Lanford-Ruelle Characterization of Gibbs States

What is missing in the heuristic proof of the Cameron-Martin formula to be rigorous? The difficulty is that the Brownian trajectories w are so rough that $Q(w)$ is infinite if calculated as the limit of the Riemann sums $\sum_{i=1}^{N}(\Delta\, w_i)^2/\Delta\, t_i$, where

$$0 = t_0 < t_1 < \ldots < t_N = 1, \quad \Delta t_i = t_i - t_{i-1}, \quad \Delta\, w_i = w(t_i) - w(t_{i-1}).$$

However, in the Cameron-Martin formula we need only the difference $Q(w) - Q(w-\varphi)$. The infinite part drops out in the difference provided φ is smooth enough, e.g. if φ is of class C^2 on $T = [0,1]$. A similar situation occurs in statistical mechanics in the case of infinite volumes. For a configuration w, the formal Hamiltonian $H(w)$ may be infinite. But if a configuration w' is obtained by a local modification of w, by changing the states in finitely many sites, then $H(w) - H(w')$ is finite. This is the strategy underlying the Dobrushin-Lanford-Ruelle (DLR) characterization of Gibbs states [2].

1.4 Malliavin Formula

According to the Malliavin formula,

$$\int_{\mathbb{W}} d\gamma(w)\ D_\varphi F(w) = \int_{\mathbb{W}} d\gamma(w)\ A_\varphi(w) F(w), \qquad (16)$$

where $D_\varphi F$ is the Gateaux differential of F in the φ-direction,

$$D_\varphi F(w) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[F(w+\epsilon\varphi) - F(w)\right] = \int_T dt\ \frac{\delta F(w)}{\delta w(t)}\ \varphi(t)\ , \qquad (17)$$

and where

$$\begin{aligned} A_\varphi(w) &:= \int_T dw(t)\ \dot\varphi(t) \\ &= -\int_T dt\ w(t)\ \ddot\varphi(t) \qquad (18) \end{aligned}$$

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for $\varphi \in C^2(0,1)$, with $\varphi(0) = \dot{\varphi}(1) = 0$. Again, the Malliavin formula is "obvious" if we use the formal expression (6) and integrate the left hand side of (16) by parts,

$$\int_{\mathbb{W}} \mathcal{D}w \ \exp\left(-\pi Q(w)\right) \ D_\varphi F(w) = -\int_{\mathbb{W}} \mathcal{D}w \ D_\varphi(\exp\left(-\pi Q(w)\right)) \ F(w) \ . \tag{19}$$

The Cameron-Martin formula and its infinitesimal form, the Malliavin formula, pave the way for defining a formal translation-invariant symbol "$\mathcal{D}$". In this paper, we propose an infinitesimal characterization of $\mathcal{D}$; namely, given an arbitrary functional U integrable by $\mathcal{D}w$, the translation invariance of $\mathcal{D}$ can be expressed by integrating by parts

$$\int_{\mathbb{W}} \mathcal{D}w \frac{\delta U}{\delta w(t)} = -\int_{\mathbb{W}} \frac{\delta}{\delta w(t)} \mathcal{D}w \cdot U = 0 \qquad \forall\, U \ . \tag{20}$$

1.5 Some Lessons from the Malliavin Formula

- Let us write formally

$$d\gamma(w) = \mu(w)\, \mathcal{D}w, \tag{21}$$

where $\mu(w)$, often called "measure" in physics, is not necessarily $\exp\left(-\pi Q(w)\right)$. Malliavin's formula reads

$$\int_{\mathbb{W}} \mu(w)\, \mathcal{D}w\, \left(D_\varphi F(w) - A_\varphi(w)\, F(w)\right) = 0, \tag{22}$$

which, by a formal integration by parts, becomes

$$\int_{\mathbb{W}} \mathcal{D}w\, \left(D_\varphi \mu(w) + A_\varphi(w)\, \mu(w)\right) \ F(w) = 0 \ . \tag{23}$$

Since $F(w)$ is arbitrary, Malliavin's formula is equivalent to

$$D_\varphi \mu(w) + A_\varphi(w)\, \mu(w) = 0 \tag{24}$$

for all φ sufficiently regular, e.g. $\varphi \in C^2(0,1)$. The "measure" $\mu(w)$ is, modulo a multiplicative constant, characterized by (24). If $\mu(w) = \exp\left(-\pi Q(w)\right)$ as before, then

$$D_\varphi Q(w) = \frac{1}{\pi} A_\varphi(w). \tag{25}$$

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According to (18), $A_\varphi(w)$ is bilinear in φ and w; hence Q is quadratic in w and we recover formula (13).

- The above remark is a heuristic proof that the Wiener measure γ is characterized by Malliavin's formula. The proof can be made rigorous by choosing in (16)

$$F(w) = \exp\left(-2\pi i \langle w', w\rangle\right) \ , \qquad w' \in \mathbb{W}' \ .$$

- Conversely the Cameron-Martin formula provides a rigorous proof of the Malliavin formula (16): replace φ by $\epsilon\varphi$ in (3) and take the derivative of both sides with respect to ϵ. At $\epsilon = 0$, one checks that

$$\frac{d}{d\epsilon} J(\epsilon\varphi, w)\bigg|_{\epsilon=0} = A_\varphi(w) \ .$$

- The Malliavin formula can be used for realizing creation and annihilation operators on bosonic Fock spaces [3], thanks to the Wiener chaos isomorphism: Let $\mathcal{H} := L^{2,1}(T)$ be a one-particle (real) Hilbert space with scalar product $(\varphi_1|\varphi_2) = \int_T dt\ \dot\varphi_1(t)\,\dot\varphi_2(t)$. Let $\mathcal{F}(\mathcal{H})$ be the Fock space with vacuum Ω. The *Wiener chaos* is an isomorphism (see e.g. Ref. [4])

$$L^2(\mathbb{W}, d\gamma) \simeq \mathcal{F}(\mathcal{H})\,. \tag{26}$$

With

$$A_\varphi(w) := \int_T dw(t)\ \dot\varphi(t),$$

one obtains

$$\int_{\mathbb{W}} d\gamma(w)\ A_{\varphi_1}(w)\ A_{\varphi_2}(w) = \int_T dt\ \dot\varphi_1(t)\ \dot\varphi_2(t). \tag{27}$$

It suffices to consider the case $\varphi_1 = \varphi_2 = \varphi$. This fundamental formula can then be established by a linear change of variable of integration in $\mathbb{W}$,

$$A_\varphi : w \longmapsto \int_T dw(t)\ \dot\varphi(t).$$

Equation (27) says that the map from $\mathcal{H}$ to $\mathcal{F}$,

$$\mathcal{H} := L^{2,1}(T) \longrightarrow \mathcal{F} := L^2\left(\mathbb{W}, d\gamma\right) \tag{28}$$

by

$$\varphi \longmapsto A_\varphi \tag{29}$$

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is an isometry. The Malliavin formula (16), with $\varphi = \varphi_1$ and $F = A_{\varphi_2}$, gives another proof of (27),

$$(\varphi_1|\varphi_2) = \int_{\mathbb{W}} d\gamma(w)\ D_{\varphi_1}(A_{\varphi_2}(w)) = \int_{\mathbb{W}} d\gamma(w)\ A_{\varphi_1}(w)\ A_{\varphi_2}(w), \quad (30)$$

and the following commutation relations are obvious:

$$[D_{\varphi_1}\ ,\ D_{\varphi_2}] = 0, \quad [A_{\varphi_1}\ ,\ A_{\varphi_2}] = 0, \quad [D_{\varphi_1}\ ,\ A_{\varphi_2}] = (\varphi_1|\varphi_2)\,. \quad (31)$$

Therefore

$$a^\dagger(\varphi) := A_\varphi - D_\varphi, \quad a(\varphi) := D_\varphi \quad (32)$$

obey the bosonic commutation relation of creation and annihilation operators on $\mathcal{F}$, respectively,

$$\left[a(\varphi_1)\ ,\ a^\dagger(\varphi_2)\right] = (\varphi_1|\varphi_2)\cdot \mathbb{1},$$

other commutators vanishing. It can be proved that a and $a^\dagger$ are adjoint in the Hilbert space $L^2(\mathbb{W}, d\gamma)$ by integrating by parts the Malliavin formula (16) with $F = F_1\, F_2$,

$$\int_{\mathbb{W}} d\gamma\ D_\varphi F_1 \cdot F_2 = \int_{\mathbb{W}} d\gamma\ F_1 \cdot (A_\varphi F_2 - D_\varphi F_2)\,. \quad (33)$$

The vacuum $\Omega \in \mathcal{F}$ is the constant function equal to 1. If a functional F of the Brownian path w acts on $\mathcal{F}$ by multiplication, i.e.

$$F(w) : \Psi(w) \longmapsto F(w)\,\Psi(w),$$

then we derive the tautology

$$\langle\Omega|F|\Omega\rangle = \int_{\mathbb{W}} d\gamma(w)\ F(w). \quad (34)$$

The Wiener measure is therefore the spectral measure corresponding to the vacuum state Ω. The vacuum is characterized by

$$a(\varphi)\,\Omega = D_\varphi \Omega = 0, \quad \forall\, \varphi$$

or alternatively by

$$\frac{\delta\,\Omega(w)}{\delta\, w(t)} = 0, \quad \forall t \in T.$$

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With the notation of (34), we write

$$0 = \langle F | a(\varphi)\, \Omega \rangle = \langle a^\dagger(\varphi)\, F | \Omega \rangle .$$

Hence the vacuum Ω is, up to a scalar, the unique state orthogonal to all $a^\dagger(\varphi)F$, that is to the functions $D_\varphi F - A_\varphi \cdot F$. This gives another interpretation to Malliavin's formula (16).

To show that $\left\{ L^2(\mathbb{W}, d\gamma)\,,\, \Omega,\, a(\varphi),\, a^\dagger(\varphi) \right\}$ is indeed a model of Fock space, it remains to check that the vectors $a^\dagger(\varphi_1) \ldots a^\dagger(\varphi_n)\, \Omega$ make a total system for $L^2(\mathbb{W}, d\gamma)$, i.e. that the finite linear combinations of $A_{\varphi_1} \ldots A_{\varphi_n}$ are dense in $L^2(\mathbb{W}, d\gamma)$. The Wiener chaos follows from the general theory of Fock space. The general theory includes not only the symmetric space considered here, but also the antisymmetric Fock space which we have not yet considered.

1.6 Feynman Volume Form

The Fourier transform of the Wiener measure (1) or (8) suggests a characterization of the Feynman volume form by its Fourier transform (see Refs. [5,6]). Let $s \in \{1, i\}$, then we can define $\mathcal{D}x$ by

$$\int_{\mathbb{X}} \mathcal{D}x \, \exp\left(-\frac{\pi}{s} Q(x) \right) \, \exp\left(-2\pi i \langle x', x \rangle \right) = \exp\left(-s\pi W(x') \right) , \qquad (35)$$

where $\mathbb{X}$ is the space of paths x and the quadratic form

$$Q(x) > 0 \text{ for } s = 1, \quad \operatorname{Im} Q(x) > 0 \text{ for } s = i \, . \qquad (36)$$

The case $s = 1$ corresponds to the Wiener measure, while the case $s = i$ corresponds to the Feynman sum over paths in quantum mechanics. Everything said before can be repeated with obvious changes, e.g. the Malliavin formula.

2 Volume Forms in Quantum Field Theory; Schwinger's Dynamical Principle

The functional integral representation of the Schwinger dynamical principle has led Bryce DeWitt to the introduction of a ubiquitous volume form in quantum field theory. According to Schwinger, the variation of the probability amplitude for a transition $\langle \text{out} | \text{in} \rangle$ is given by the variation of the action $\mathbf{S}$ of

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the system[c]:

$$\delta \langle \text{out} \,|\, \text{in} \rangle = \frac{i}{\hbar} \langle \text{out} | \, \delta \mathbf{S} \, | \text{in} \rangle, \tag{37}$$

where $\mathbf{S}$ is a functional of the field operators, which are globally designated by $\boldsymbol{\varphi}$.

2.1 Evolution Equations for the Field Operators $\boldsymbol{\varphi}$

The Schwinger variational principle gives evolution equations for the field operators $\boldsymbol{\varphi}$ different from the classical Euler-Lagrange equations

$$\frac{\delta S}{\delta \varphi} = 0 \,. \tag{38}$$

The Schwinger-Dyson equations give the quantum evolution of polynomials of fields $F(\boldsymbol{\varphi})$ for a system with classical action S by the expectation value of a time ordered operator,[d]

$$\langle \text{vac} | \, T \left(\frac{i}{\hbar} \frac{\delta S}{\delta \boldsymbol{\varphi}} F(\boldsymbol{\varphi}) + \frac{\delta F}{\delta \boldsymbol{\varphi}} \right) | \text{vac} \rangle = 0 \,. \tag{39}$$

2.2 Functional Integral Solution of the Schwinger Principle

To exploit the Schwinger variational principle (37), one varies an external source J added to the original action S. The new action is

$$S + \langle J, \boldsymbol{\varphi} \rangle,$$

and the principle (37) now reads

$$\frac{\hbar}{i} \frac{\delta}{\delta J} \langle \text{out} \,|\, \text{in} \rangle = \langle \text{out} | \, \boldsymbol{\varphi} \, | \text{in} \rangle.$$

Bryce DeWitt has constructed the following functional integral solution of this equation (for details, see pp. 4160-4164 and related references in Ref. [6]):

$$\langle \text{out} \,|\, \text{in} \rangle = \mathcal{N} \int_{\Phi(\text{in},\text{out})} \mu(\varphi) \, \mathcal{D}\varphi \, \exp \left(\frac{i}{\hbar} (S(\varphi) + \langle J, \varphi \rangle) \right) , \tag{40}$$

where

[c] We use boldface for operators on Fock space.

[d] The proof of Eq. (39) and some of its applications can be found e.g. in the textbook by Peskin and Schroeder [7], Section 9.6.

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- $\mathcal{N}$ is a normalization constant,
- the domain of integration is defined by the *in* and *out* states,
- $\mathcal{D}\varphi$ is invariant under translations,
- $\mu(\varphi)$ is, to leading order, given by the advanced Green's function G^+:

$$\mu(\varphi) = |\text{sdet } G^+(\varphi)|^{-1/2} + \dots, \tag{41}$$

where "sdet" is the superdeterminant. In (41) the advanced Green's function G^+ is the unique inverse of the leading non-constant term S'' of the expansion of S restricted to its domain of integration $\Phi(\text{in}, \text{out})$:

$$S(\varphi_0 + \delta\boldsymbol{\varphi}) = S(\varphi_0) + \frac{1}{2} S''(\varphi_0)\delta\boldsymbol{\varphi}\, \delta\boldsymbol{\varphi} + \dots \ , \qquad S'(\varphi_0) = 0. \tag{42}$$

Equations (35) and (40) define volume forms $\mu(\varphi)\mathcal{D}\varphi$ and $\mathcal{D}x$, respectively. In both cases $\mathcal{D}$ is a translation-invariant symbol. For comparing the structure of the two volume forms, we write the finite-dimensional version of (35) with $s = 1$:

$$\int_{\mathbb{R}^D} \mathcal{D}x \ \exp(-\pi Q_{\alpha\beta} x^\alpha x^\beta) \exp(-2\pi i x'_\alpha x^\alpha) = \exp(-\pi W^{\alpha\beta} x'_\alpha x'_\beta), \tag{43}$$

where

$$Q_{\alpha\beta} W^{\beta\gamma} = \delta^\gamma_\alpha, \tag{44}$$

$$\begin{aligned} \mathcal{D}x &= (\det Q_{\alpha\beta})^{1/2}\, dx^1 \dots dx^D \\ &= (\det W^{\alpha\beta})^{-1/2}\, dx^1 \dots dx^D \ . \end{aligned} \tag{45}$$

Hence G^+ is to field theory what W is to a Gaussian on $\mathbb{R}^D$, that is, the covariance matrix:

$$W^{\lambda\mu} = 2\pi \int_{\mathbb{R}^D} \mathcal{D}x \ \exp\left(-\pi Q_{\alpha\beta} x^\alpha x^\beta\right) \ x^\lambda x^\mu.$$

In (40) the term $(2\pi i/h)\langle J, \varphi\rangle$ corresponds to $-2\pi i\langle x', x\rangle$ in (35); they both mean that the left-hand side is a Fourier transform.

In finite dimensions it is easy to write explicitly $\mathcal{D}x$; in infinite dimensions it has meaning only in the context of the integral (35). However, this implicit definition is sufficient for computing functional integrals [5,6]. It is also easy to generalize it to cases other than Gaussians.

Using a generalized formulation of $\mathcal{D}x$, or using $\mu(\varphi)\mathcal{D}\varphi$ obtained from the solution of the Schwinger variational principle is a matter of choice and

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is often dictated by the context. Once $\mu(\varphi)$, or at least its leading term, is defined, the following equation

$$\langle \text{out} \,|\, T(F(\boldsymbol{\varphi})) \,|\, \text{in} \rangle = \mathcal{N} \int \mu(\varphi)\mathcal{D}\varphi \, F(\varphi) \exp(\frac{i}{\hbar} S(\varphi))$$

can be exploited in a variety of cases, i.e. in cases where $F(\varphi)$ is not simply $\exp(i\langle J, \varphi\rangle/\hbar)$.

3 Volume Forms in Differential Geometry

We shall use differential geometry for defining volume forms on finite-dimensional Riemannian and symplectic manifolds in a formulation which paves the way for the infinite-dimensional case. The knowledgeable reader for whom using the infinite limit of a finite volume element is, rightly, anathema, please bear with us. Finite-dimensional volume elements are useful in the following situations:

- A rule of thumb. A statement which is independent of the dimension of the space of interest has a chance to generalize to infinite-dimensional spaces; for example a Gaussian on $\mathbb{R}^D$ defined by (43) generalizes easily to (8).
- Infinite-dimensional spaces defined by a projective system of finite-dimensional spaces. This strategy was used in defining Feynman volume forms by their Fourier transforms [8,9].
- Differential calculus on Banach spaces, and differential geometry on Banach manifolds. They are natural generalizations of their finite dimensional counterparts. For this reason we propose a formula which defines volume elements by their Lie derivatives.

Let $\mathcal{L}_X$ be the Lie derivative with respect to a vector field X on a D-dimensional manifold M^D, either a (pseudo-)Riemannian manifold (M^D, g) or a symplectic manifold (M^{2N}, Ω). The volume forms are, respectively,

$$\omega_g(x) = |\det g_{\alpha\beta}(x)|^{1/2} dx^1 \wedge \ldots \wedge dx^D \qquad \text{on } (M^D, g) \tag{46}$$

and

$$\omega_\Omega(x) = \frac{1}{N!} \Omega \wedge \ldots \wedge \Omega \ \ (N \text{ factors}) \qquad \text{on } (M^{2N}, \Omega). \tag{47}$$

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In canonical coordinates (p, q),

$$\Omega = \sum_{\alpha} dp_{\alpha} \wedge dq^{\alpha} \tag{48}$$

and

$$\omega_{\Omega} = dp_1 \wedge dq^1 \wedge \ldots \wedge dp_N \wedge dq^N \ . \tag{49}$$

Surprisingly ω_g and ω_{Ω} satisfy equations of the same structure:

$$\mathcal{L}_X \omega_g = \tfrac{1}{2}\mathrm{Tr}\left(g^{-1}\mathcal{L}_X g\right)\omega_g, \tag{50}$$

$$\mathcal{L}_X \omega_{\Omega} = \tfrac{1}{2}\mathrm{Tr}\left(\Omega^{-1}\mathcal{L}_X \Omega\right)\omega_{\Omega}. \tag{51}$$

Riemannian and symplectic geometry are notoriously different (see e.g. McDuff [10]) and the analogies between them are not superficial. For instance, with Riemannian geometry on the left and symplectic geometry on the right

$$\begin{array}{cc} \int ds & \int \Omega \\ \text{geodesics} & \text{minimal surfaces} \\ \mathcal{L}_X g = 0 \text{ defines} & \mathcal{L}_X \Omega = 0 \text{ defines} \\ \text{Killing vector} & \text{Hamiltonian vector fields} \end{array} \tag{52}$$

Killing vector fields are few, Hamiltonian vector fields are many.

3.1 The General Case

Before proving Eqs. (50) and (51), we consider the more general equation

$$\mathcal{L}_X \omega = D(X) \cdot \omega \tag{53}$$

or its integrated formulation

$$\int_M (\mathcal{L}_X F)\ \omega = -\int_M F\ \mathcal{L}_X \omega = -\int_M F\ D(X) \cdot \omega, \tag{54}$$

where ω is a top form (a D-dimensional form on M^D) and $D(X)$ is a function on M depending on the vector field X on M.

- Properties of $D(X)$ dictated by properties of $\mathcal{L}_X$:

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X \ \Leftrightarrow\ D([X,Y]) = X\left(D(Y)\right) - Y\left(D(X)\right) \tag{55}$$

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On a top form:

$$\mathcal{L}_{fX} = f\mathcal{L}_X + X(f) \;\Leftrightarrow\; D(fX) = fD(X) + X(f)\,. \tag{56}$$

PROOF: On a top form ω, the Cartan formula $\mathcal{L}_X = d\,i_X + i_X\,d$ yields

$$\mathcal{L}_X\omega = d\,i_X\omega\ ,$$

and since $i_{fX}\omega = i_X\,(f\omega)$, we have $\mathcal{L}_{fX}\omega = d\,i_X\,(f\omega) = \mathcal{L}_X\,(f\omega)$. ■

- In coordinates,

$$\omega_\mu(x) = \mu(x)\ dx^1 \wedge \ldots \wedge dx^D = \mu(x)\,d^Dx\ . \tag{57}$$

By the Leibniz rule,

$$\mathcal{L}_X(\mu\,d^Dx) = \mathcal{L}_X(\mu)\,d^Dx + \mu\,\mathcal{L}_X\left(d^Dx\right)\ . \tag{58}$$

Because d^Dx is a top form on M^D,

$$\mathcal{L}_X\left(d^Dx\right) = d(i_Xd^Dx) = X^\alpha{}_{,\,\alpha}\,d^Dx\ . \tag{59}$$

Finally, combining (57), (58) and (59),

$$\begin{aligned}\mathcal{L}_X\,\omega_\mu &= (X^\alpha\mu_{,\alpha} + \mu\,X^\alpha{}_{,\alpha})\,d^Dx\\ &= D(X)\cdot\omega_\mu,\end{aligned} \tag{60}$$

with

$$\begin{aligned}D(X) &= (X^\alpha\mu_{,\alpha} + \mu\,X^\alpha{}_{,\alpha})\ \mu^{-1}\\ &= X^\alpha{}_{,\alpha} + X^\alpha(\log|\mu|)_{,\alpha}\ .\end{aligned} \tag{61}$$

3.2 The Riemannian Case (M, g)

Let $\omega_g(x) = \mu(x)d^Dx$. We shall show that the basic equation (50) is satisfied if and only if $\mu(x) = \text{const}\,|\det g(x)\,|^{1/2}$. Indeed

$$(\mathcal{L}_Xg)_{\alpha\beta} = X^\gamma g_{\alpha\beta,\gamma} + g_{\gamma\beta}X^\gamma{}_{,\alpha} + g_{\alpha\gamma}X^\gamma{}_{,\beta} \tag{62}$$

and

$$\mathrm{Tr}\left(g^{-1}\mathcal{L}_Xg\right) = g^{\beta\alpha}X^\gamma g_{\alpha\beta,\gamma} + 2X^\alpha_{,\alpha} \tag{63}$$

and, as already computed in Eq. (60),

$$\mathcal{L}_X\left(\mu(x)\,d^Dx\right) = (X^\alpha\,\mu_{,\alpha} + \mu\,X^\alpha{}_{,\alpha})\ d^Dx. \tag{64}$$

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Therefore the basic equation (50) is satisfied if, and only if

$$(X^\gamma\, \mu_{,\gamma} + \mu\, X^\alpha{}_{,\alpha})\ \mu^{-1} = \frac{1}{2}\left(g^{\beta\alpha} X^\gamma g_{\alpha\beta,\gamma} + 2\, X^\alpha{}_{,\alpha}\right), \tag{65}$$

i.e.

$$\frac{\mu_{,\gamma}}{\mu} = \frac{1}{2}\, g^{\beta\alpha} g_{\alpha\beta,\gamma} = \frac{1}{2}\,\partial_\gamma \ln|\det g\,|, \tag{66}$$

$$\mu(x) = \text{ const. } |\det g(x)\,|^{1/2}. \tag{67}$$

The equation

$$\mathcal{L}_X \omega_g = \frac{1}{2}\mathrm{Tr}\left(g^{-1}\mathcal{L}_X g\right)\,\omega_g \tag{68}$$

has, up to multiplication by a constant, a unique solution

$$\omega_g(x) = |\det g(x)|^{1/2} dx^1 \wedge \ldots \wedge dx^D\ . \qquad \blacksquare \tag{69}$$

We use the classical formula

$$\Gamma^\alpha{}_{\alpha\gamma} = \frac{1}{2}\, g^{\beta\alpha} g_{\alpha\beta,\gamma}, \tag{70}$$

with the Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$. Hence (63) says

$$\frac{1}{2}\mathrm{Tr}\left(g^{-1}\,\mathcal{L}_X g\right) = X^\alpha{}_{;\alpha} =: \mathrm{Div}_g(X) \tag{71}$$

with the standard definition of the covariant divergence $X^\alpha{}_{;\alpha}$ of the vector field X, and we can write the basic equation (50) in the form

$$\mathcal{L}_X \omega_g = \mathrm{Div}_g(X)\cdot\omega_g\ . \tag{72}$$

If X is a Killing vector field with respect to isometries, then $\mathcal{L}_X g = 0$, $\mathcal{L}_X \omega_g = 0$ and $X_{\alpha\,;\,\beta} + X_{\beta\,;\,\alpha} = 0$. Hence $X^\alpha{}_{;\alpha} = 0$ and (72) is satisfied.

3.3 The Symplectic Case (M^D, Ω), $D = 2N$

The symplectic form Ω on M^{2N} is a closed 2-form of rank $D = 2N$.

$$\begin{aligned}
\Omega &= \Omega_{AB}\, dx^A \wedge dx^B && \text{with } A < B,\ \Omega_{AB} = -\Omega_{BA},\ d\,\Omega = 0 \\
&= \tfrac{1}{2}\,\Omega_{\alpha\beta}\, dx^\alpha \wedge dx^\beta && \text{no restriction on the order of } \alpha,\ \beta \\
&= \tfrac{1}{2}\,\Omega_{\alpha\beta}(dx^\alpha \otimes dx^\beta - dx^\beta \otimes dx^\alpha) \\
&= \Omega_{\alpha\beta}\, dx^\alpha \otimes dx^\beta,
\end{aligned} \tag{73}$$

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since $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$.

REMARK: There are two different definitions of the exterior product, each with its concomitant definition of exterior derivative, e.g.

$$dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \otimes dx^1, \tag{74}$$

$$\tilde{d}x^1 \tilde{\wedge} \tilde{d}x^2 = \frac{1}{2}\left(\tilde{d}x^1 \otimes \tilde{d}x^2 - \tilde{d}x^2 \otimes \tilde{d}x^1\right). \tag{75}$$

With the second definition, Stokes' formula for a p-form θ reads $\int_M \tilde{d}\theta = (p+1)\int_{\partial M}\theta$; with the first one, it is simply $\int_M d\theta = \int_{\partial M}\theta$. We choose the first definition, namely

$$df^1 \wedge \ldots \wedge df^p = \epsilon_{j_1 \ldots j_p}\, df^{j_1} \otimes \ldots \otimes df^{j_p},$$

and in particular

$$dx^1 \wedge \ldots \wedge dx^D = \epsilon_{j_1 \ldots j_D}\, dx^{j_1} \otimes \ldots \otimes dx^{j_D},$$

where ϵ is totally antisymmetric. Since Ω is of rank $D = 2N$,

$$\Omega^{\wedge N} := \Omega \wedge \ldots \wedge \Omega \qquad (N \text{ factors})$$

is a nonzero top form on M^{2N} and the volume element

$$\omega_\Omega = \frac{1}{N!}\,\Omega^{\wedge N} \tag{76}$$

$$= \mathrm{Pf}(\Omega_{\alpha\beta})\, d^D x = |\det \Omega_{\alpha\beta}|^{1/2}\, d^D x\ . \tag{77}$$

We shall show that the basic equation (51) is satisfied if and only if ω_Ω is proportional to the volume form (76).

PROOF: We define the inverse Ω^{-1} of Ω, calculate the quantity $\frac{1}{2}\mathrm{Tr}\left(\Omega^{-1}\mathcal{L}_X\Omega\right)$, then prove the basic formula (51).

- The symplectic form Ω defines an isomorphism from the tangent bundle TM to the cotangent bundle T^*M by

  $$\Omega : X \longmapsto i_X\Omega.$$

  We can then define

  $$X_\alpha := X^\beta \Omega_{\beta\alpha}. \tag{78}$$

  The inverse $\Omega^{-1} : T^*M \longrightarrow TM$ is given by

  $$X^\alpha = X_\beta\, \Omega^{\beta\alpha},$$

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with

$$\Omega^{\alpha\beta}\,\Omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}. \tag{79}$$

Note that in strict components, i.e. with $\Omega = \Omega_{AB}\, dx^A \wedge dx^B$ with $A < B$, X_A is not equal to $X^B \Omega_{BA}$.

- We compute

$$\begin{aligned}(\mathcal{L}_X\Omega)_{\alpha\beta} &= X^{\gamma}\,\Omega_{\alpha\beta,\gamma} + \Omega_{\gamma\beta}\,X^{\gamma}_{\ ,\alpha} + \Omega_{\alpha\gamma}\,X^{\gamma}_{\ ,\beta} \\ &= X_{\beta,\alpha} - X_{\alpha,\beta}\end{aligned} \tag{80}$$

using $d\Omega = 0$, that is $\Omega_{\beta\gamma,\alpha} + \Omega_{\gamma\alpha,\beta} + \Omega_{\alpha\beta,\gamma} = 0$. Hence

$$\left(\Omega^{-1}\mathcal{L}_X\Omega\right)^{\gamma}{}_{\beta} = \Omega^{\gamma\alpha}\,(X_{\beta,\alpha} - X_{\alpha,\beta})$$

and

$$\frac{1}{2}\mathrm{Tr}\left(\Omega^{-1}\mathcal{L}_X\Omega\right) = \Omega^{\gamma\alpha}X_{\gamma,\alpha}. \tag{81}$$

- According to Darboux' theorem, there is a coordinate system (x^{α}) in which the volume form $\omega_{\Omega} = \Omega^{\wedge N}/N!$ is

$$\omega_{\Omega} = dx^1 \wedge \ldots \wedge dx^{2N}, \tag{82}$$

and $\Omega = \Omega_{\alpha\beta}\, dx^{\alpha} \otimes dx^{\beta}$ with *constant* coefficients $\Omega_{\alpha\beta}$. The inverse matrix $\Omega^{\beta\alpha}$ is also made of constants, hence $\Omega^{\beta\alpha}{}_{,\gamma} = 0$.
In these coordinates

$$\begin{aligned}\mathcal{L}_X\omega_{\Omega} &= X^{\alpha}{}_{,\alpha}\,\omega_{\Omega} = \left(X_{\beta}\,\Omega^{\beta\alpha}\right)_{,\alpha}\omega_{\Omega} \\ &= \left(X_{\beta,\alpha}\,\Omega^{\beta\alpha} + X_{\beta}\,\Omega^{\beta\alpha}{}_{,\alpha}\right)\omega_{\Omega} = X_{\beta,\alpha}\,\Omega^{\beta\alpha}\omega_{\Omega}\end{aligned} \tag{83}$$

and we conclude by using (81). ■

If X is a Hamiltonian vector field, then $\mathcal{L}_X\Omega = 0$ and $\mathcal{L}_X\omega_{\Omega} = 0$. The basic equation (51) is trivially satisfied.

4 Conclusion

Integration by parts is the key to the progress made in this paper for characterizing volume forms. It makes possible an infinitesimal characterization of the translation invariant symbol $\mathcal{D}$,

$$\int \mathcal{D}\varphi\,\frac{\delta U}{\delta\varphi(x)} = 0, \tag{84}$$

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(World Scientific, Singapore, 2001).

and is more powerful than its global translation (7),

$$\mathcal{D}(\varphi + \varphi_0) - \mathcal{D}\varphi = 0\ . \tag{85}$$

The challenges we are now considering are the following:

- to extend to infinite-dimensional spaces the divergence formulae (50) and (51).
- to clarify the often observed relationship between the volume form and the Schrödinger equation satisfied by a functional integral.
- to develop issues mentioned briefly in this paper, in particular the Dobrushin-Lanford-Ruelle formula, and the annihilation/creation operators defined by the Malliavin formula.
- to derive the transformation laws of volume elements under the Cartan development mapping between two spaces of pointed paths on different manifolds.
- to extend the method from ordinary (bosonic) integration to Berezin (fermionic) integration.

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