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COORDINATE INDEPENDENCE OF PATH INTEGRALS IN ONE-DIMENSIONAL TARGET SPACE

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The recent perturbative formulation of the quantum mechanical path integral in a curved space allows one to calculate the Feynman diagrams involving multiple temporal integrals over products of distributions. To test the consistency of this formulation the exactly solvable path integral for the particle in a flat space is transformed into the curvilinear form of a nonlinear sigma model where the only solution is the perturbation expansion. By an explicit three-loop calculation we show that the perturbatively defined path integral in new curvilinear coordinates reproduces the ground-state energy of the original system thus ensuring the coordinate independence.

1 Introduction

In honor of Professor Kleinert's birthday I would like to review some of the most interesting results we found in a series of recent papers written in a common collaboration [1–3].

By an arbitrary coordinate transformation from flat to curved space the exactly solvable path integral for the particle in a flat space can be turned into the curvilinear form of a nonlinear sigma model where the only solution is the perturbation expansion. The formal perturbative definition of quantum mechanical path integrals in curvilinear coordinates, however, poses problems. To exhibit the difficulties, consider the associated partition function calculated for periodic paths on the imaginary-time axis τ:

$$Z = \int \mathcal{D}q(\tau)\sqrt{g}\, e^{-\mathcal{A}[q]}, \tag{1}$$

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where $\mathcal{A}[q]$ is the Euclidean action with the general form

$$\mathcal{A}[q] = \int d\tau \left[\frac{1}{2} g_{\mu\nu}(q(\tau)) \dot{q}^{\mu}(\tau) \dot{q}^{\nu}(\tau) + V(q(\tau)) \right]. \tag{2}$$

The dots denote τ-derivatives, $g_{\mu\nu}(q)$ is a metric, and $g = \det g_{\mu\nu}$ its determinant. The path integral may formally be defined perturbatively as follows: The metric $g_{\mu\nu}(q)$ is expanded around some point q_0^{μ} in powers of the deviation $\delta q^{\mu} \equiv q^{\mu} - q_0^{\mu}$. The same thing is done with the potential $V(q)$. After this, the action $\mathcal{A}[q]$ is separated into a free part $\mathcal{A}_0[q_0; \delta q] \equiv g_{\mu\nu}(q_0)\dot{q}^{\mu}\dot{q}^{\nu}/2 + \omega^2 \delta q^{\mu} \delta q^{\nu}/2$, and an interacting part $\mathcal{A}_{\rm int}[q_0; \delta q] \equiv \mathcal{A}[q] - \mathcal{A}_0[q_0; \delta q]$. The square root in the path integral (1) is taken into the exponent and expanded likewise, defining an effective action $\mathcal{A}_{\sqrt{g}} = -\delta(0) \log[g(q_0 + \delta q)/g(q_0)]/2$, where $\delta(0)$ is the δ-function at the origin. It represents formally the inverse infinitesimal lattice spacing on the time axis, and is equal to $\delta(0) \equiv \int dp/(2\pi)$.

An expansion of Z in powers of the interaction leads to a sum of loop diagrams. Problems arise from the fact that there are interaction terms involving $\dot{q}^2 q^n$ which lead to Feynman integrals over products of distributions. There are now three types of lines representing the correlation functions:

$$\Delta(\tau, \tau') \equiv \langle q(\tau) q(\tau') \rangle = \text{———}\,, \tag{3}$$

$$\partial_{\tau} \Delta(\tau, \tau') \equiv \langle \dot{q}(\tau) q(\tau') \rangle = \text{-----———}\,, \tag{4}$$

$$\partial_{\tau} \partial_{\tau'} \Delta(\tau, \tau') \equiv \langle \dot{q}(\tau) \dot{q}(\tau') \rangle = \text{-------}\,. \tag{5}$$

The right-hand sides define the line symbols to be used in Feynman diagrams for the interaction terms.

Explicitly, the first correlation function reads

$$\Delta(\tau, \tau') = \frac{1}{2\omega} e^{-\omega|\tau - \tau'|}. \tag{6}$$

The second correlation function (4) has a discontinuity

$$\partial_{\tau} \Delta(\tau, \tau') = -\frac{1}{2} \epsilon(\tau - \tau') e^{-\omega|\tau - \tau'|}, \tag{7}$$

where

$$\epsilon(\tau - \tau') \equiv -1 + 2 \int_{-\infty}^{\tau} d\tau'' \delta(\tau'' - \tau') \tag{8}$$

is a distribution which vanishes at the origin and is equal to ± 1 for positive and negative arguments, respectively. The third correlation function (5)

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contains a δ-function:

$$\partial_\tau \partial_{\tau'} \Delta(\tau, \tau') = \delta(\tau - \tau') - \frac{\omega}{2} e^{-\omega|\tau-\tau'|}. \tag{9}$$

Recently, the temporal integrals over products of such distributions were defined uniquely and independently of the choice of coordinates [1–3]. In Ref. [1] it was shown that Feynman integrals in *momentum space* can be uniquely defined as $\epsilon \to 0$-limits of $(1-\epsilon)$-dimensional integrals via an analytic continuation à la 't Hooft and Veltman [4]. This definition makes path integrals coordinate-independent. In Ref. [2] the rules for the calculation of temporal integrals over products of distributions were formulated directly for Feynman integrals in the $(1-\epsilon)$-dimensional time space. In Ref. [3] these integrals were expressed uniquely via integrals over products of nonsingular correlation functions $\Delta(\tau, \tau')$, plus integrals over pure products of δ-functions. The rules defining the integrals over distributional products were set up directly in one dimension by the requirement of coordinate invariance of the path integral (1). This approach has the important advantage to make superfluous the somewhat tedious analytic continuation to $1-\epsilon$ dimensions. In fact, it does not require specifying any regularization scheme. In addition, it gives a foundation of a new and general mathematics of extending the theory of distributions from a linear space to products.

In this note, using the results of the previous work [1–3], we test the coordinate independence of perturbatively defined path integrals by an explicit three-loop calculation of the ground-state energy.

2 Model System

We want to test the coordinate independence of the exactly solvable path integral of a point particle of unit mass in a harmonic potential $\omega^2 x^2/2$, over a large imaginary-time interval β,

$$Z_\omega = \int \mathcal{D}x(\tau)\, e^{-\mathcal{A}_\omega[x]} = e^{-\mathrm{Tr}\log(-\partial^2+\omega^2)} = e^{-\beta\omega/2}. \tag{10}$$

The action is

$$\mathcal{A}_\omega = \frac{1}{2} \int d\tau \left[\dot{x}^2(\tau) + \omega^2 x^2(\tau)\right]. \tag{11}$$

Here and in the following, the target space is assumed to be one-dimensional for simplicity.

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A coordinate transformation turns (10) into a path integral of the type (1) with a singular perturbation expansion.

From our work in Refs. [1,2] we know that all terms in this expansion vanish in dimensional regularization. Here we shall use the identities for integrals over products of distributions found in Ref. [3] to perform the perturbative calculation directly in one dimension.

For simplicity we assume the coordinate transformation to preserve the symmetry $x \leftrightarrow -x$ of the initial oscillator, such that its power series expansion starts out like $x(\tau) = f(q(\tau)) = q - gq^3/3 + g^2aq^5/5 - \cdots$, where g is a smallness parameter and a an extra parameter. We shall see that the identities are independent of a, such that a will merely serve to check the calculations. The transformation changes the partition function (10) into

$$Z = \int \mathcal{D}q(\tau)\, e^{-\mathcal{A}_J[q]} e^{-\mathcal{A}[q]}, \tag{12}$$

where $\mathcal{A}[q]$ is the transformed action, whereas $\mathcal{A}_J[q]$ represents an effective action coming from the Jacobian of the coordinate transformation:

$$\mathcal{A}_J[q] = -\delta(0) \int d\tau \,\log \frac{\delta f(q(\tau))}{\delta q(\tau)}. \tag{13}$$

Of course, the quantum mechanics is the UV-finite one-dimensional quantum field theory. Up to the three-loop level, this was shown in Refs. [1,2] via an analytic continuation to $1-\epsilon$ dimensions in the $\epsilon \to 0$ -limit. Thus, working in one dimension directly, we can ignore the intermediate linear and quadratic divergences arising from the loop diagrams. Instead, we want to trace out explicitly the cancellation of these divergences against those coming from the Jacobian action (13).

The transformed action is decomposed into a free part

$$\mathcal{A}_\omega[q] = \frac{1}{2} \int d\tau [\dot{q}^2(\tau) + \omega^2 q^2(\tau)], \tag{14}$$

and an interacting part, which reads to second order in g:

$$\begin{aligned}\mathcal{A}_{\rm int}[q] = \frac{1}{2} \int d\tau \Bigg\{ &-g \left[2\dot{q}^2(\tau) q^2(\tau) + \frac{2\omega^2}{3} q^4(\tau) \right] \\ &+g^2 \left[(1+2a)\, \dot{q}^2(\tau) q^4(\tau) + \omega^2 \left(\frac{1}{9} + \frac{2a}{5} \right) q^6(\tau) \right] \Bigg\}. \end{aligned} \tag{15}$$

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To the same order in g, the Jacobian action (13) is

$$\mathcal{A}_J[q] = -\delta(0) \int d\tau \left[-g q^2(\tau) + g^2 \left(a - \frac{1}{2} \right) q^4(\tau) \right]. \tag{16}$$

For $g = 0$, the transformed partition function (12) coincides with (10). When expanding Z of Eq. (12) in powers of g, we obtain sums of Feynman diagrams contributing to each order g^n, which must vanish to ensure coordinate invariance. By considering only connected Feynman diagrams, we are dealing directly with the ground-state energy.

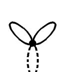
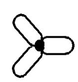
3 Free Energy Density

The graphical expansion for the ground-state energy will be carried here only up to three loops. At any order g^n, there exist different types of Feynman diagrams with $L = n+1, n,$ and $n-1$ number of loops coming from the interaction terms (15) and (16), respectively. The diagrams are composed of the three types of lines in Eqs. (3)–(5), and of new interaction vertices for each power of g. The diagrams coming from the Jacobian action (16) are easily recognized by an accompanying power of $\delta(0)$.

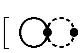
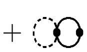
At first order in g, there exist only three diagrams, two originated from the interaction (15), one from the Jacobian action (16):

$-g$  $-g\,\omega^2$  $+g\,\delta(0)$  . (17)

At order g^2, we distinguish several contributions. First there are two three-loop local diagrams coming from the interaction (15), and one two-loop local diagram from the Jacobian action (16):

$g^2 \Big[3\left(\frac{1}{2}+a\right)$  $+ 15\omega^2 \left(\frac{1}{18}+\frac{a}{5}\right)$ 
$- 3\left(a-\frac{1}{2}\right)\delta(0)$  $\Big]$. (18)

We call a diagram local if it involves no temporal time integral. The Jacobian action (16) contributes further the nonlocal diagrams:

$-\frac{g^2}{2!}\Big\{ 2\delta^2(0)$  $- 4\delta(0)\big[$  $+$  $+ 2\,\omega^4$  $\big]\Big\}$. (19)

The remaining diagrams come from the interaction (15) only. They are either of the three-bubble type, or of the watermelon type, each with all possible

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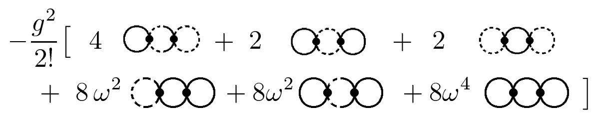
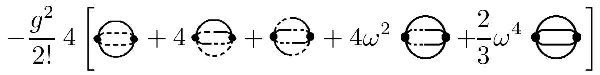
combinations of the three-line types (3)–(5): The sum of all three-bubbles diagrams is



(20)

while the watermelon-like diagrams contribute



(21)

Since the equal-time expectation value $\langle \dot{q}(\tau)\, q(\tau) \rangle$ vanishes according to Eq. (7), there are a number of trivially vanishing diagrams, which have been omitted.

In previous papers [1,2], all integrals were calculated individually in $D = 1 - \varepsilon$ dimensions, taking the limit $\varepsilon \to 0$ at the end. The results for the integrals ensured that the sum of all Feynman diagrams contributing to each order g^n vanishes.

4 Integrals over Distributions

Here we shall follow the developments of Ref. [3] for calculating temporal integrals over products of distributions directly in one dimension. These integrals can be expressed formally via integrals over products of nonsingular correlation functions $\Delta(\tau, \tau')$, plus integrals containing pure products of δ- and ϵ-functions.

First, we shall calculate directly some basic integrals. Most simply, we obtain the relations

$$\int d\tau\, \Delta^2(\tau) = \frac{1}{2\omega^2}\Delta(0), \tag{22}$$

$$\int d\tau\, \dot{\Delta}^2(\tau) = \frac{1}{2}\Delta(0). \tag{23}$$

Therefore we have

$$\int d\tau\, \left[\dot{\Delta}^2(\tau) + \omega^2\Delta^2(\tau)\right] = \Delta(0). \tag{24}$$

Two other elementary integrals read

$$\int d\tau \Delta^4(\tau) = \frac{\Delta^3(0)}{4\omega^2}, \tag{25}$$

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$$\int d\tau\,\dot{\Delta}^2(\tau)\Delta^2(\tau) = \frac{1}{4}\Delta^3(0). \tag{26}$$

Note that these relations can easily be obtained with the help of partial integration and the inhomogeneous field equation satisfied by the correlation function

$$\ddot{\Delta}(\tau) = -\int \bar{d}k\, \frac{k^2}{k^2+\omega^2} e^{ik\tau} = -\,\delta(\tau) + \omega^2\Delta(\tau)\,. \tag{27}$$

To derive, for example, Eq. (24) we have first to integrate the first term partially,

$$\int d\tau\,\dot{\Delta}^2(\tau) = -\int d\tau\,\Delta(\tau)\ddot{\Delta}(\tau), \tag{28}$$

with no boundary term due to the exponential vanishing at infinity of all functions involved. Using then the field equation (27) and the definition of the δ-function

$$\int d\tau\, f(\tau)\,\delta(\tau) = f(0), \tag{29}$$

we obtain (24).

The same is valid for Eq. (26): by a partial integration, the left-hand side becomes

$$\int d\tau\,\dot{\Delta}^2(\tau)\Delta^2(\tau) = -\frac{1}{3}\int d\tau\,\ddot{\Delta}(\tau)\Delta^3(\tau). \tag{30}$$

Applying again the field equation (27) and Eq. (25), we reduce the right-hand side to

$$-\int d\tau\,\ddot{\Delta}(\tau)\Delta^3(\tau) = \Delta^3(0) - \omega^2\int d\tau\,\Delta^4(\tau) = \frac{3}{4}\Delta^3(0). \tag{31}$$

Substituting this into Eq. (30), yields (26).

Consider now more singular integrals over $\ddot{\Delta}^2(\tau)$, $\ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau)$ and $\ddot{\Delta}^2(\tau)\Delta^2(\tau)$ involving pure products of δ- and ϵ-functions. As a first step in calculating these integrals we replace $\ddot{\Delta}(\tau)$ by the left-hand side of Eq. (27). This reduces the integral involving $\ddot{\Delta}^2(\tau)$ to the relation

$$\int d\tau\Big[\ddot{\Delta}^2(\tau) + 2\omega^2\dot{\Delta}^2(\tau) + \omega^4\Delta^2(\tau)\Big] = \int d\tau\,\delta^2(\tau)\,. \tag{32}$$

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Combining the same tool with Eq. (26), we rewrite the integral over $\ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau)$ in the following form

$$\int d\tau\, \ddot{\Delta}(\tau)\dot{\Delta}^2(\tau)\Delta(\tau) = -\frac{1}{8\omega}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{4}\omega^2\Delta^3(0)\,. \tag{33}$$

Let us turn now to the integral over $\ddot{\Delta}^2(\tau)\Delta^2(\tau)$. This can be expressed in terms of the regular integral

$$\int d\tau\, \dot{\Delta}^4(\tau) = \frac{1}{4}\omega^2\Delta^3(0) \tag{34}$$

as follows: Invoking once more the field equation (27) and Eq. (34), we find

$$\int d\tau\, \ddot{\Delta}^2(\tau)\Delta^2(\tau) = \int d\tau\, \Delta^2(\tau)\delta^2(\tau) - \frac{7}{4}\omega^2\Delta^3(0)\,. \tag{35}$$

As in standard quantum field theory, we should define now the integrals over distributional products of δ- and ϵ-functions, occurring in Eqs. (32), (33) and (35). Before coming to this, however, we note that the partial integration rule for integrals of this kind is not generally allowed in the purely one-dimensional approach. To illustrate the problem involved with partial integration, consider the integral

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|}, \tag{36}$$

with an arbitrary parameter λ. Applying partial integration reduces this to the completely regular form

$$3\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|} = \frac{\lambda}{2}\int d\tau\, \epsilon^4(\tau)e^{-\lambda|\tau|} = 1, \tag{37}$$

for any value of λ. Taking Eq. (34) into account, it is easy to see that the result (37) satisfies the identity

$$\begin{aligned}\int d\tau\, \Delta(\tau)\dot{\Delta}^2(\tau)\ddot{\Delta}(\tau) &= -\frac{1}{3}\int d\tau\, \dot{\Delta}^4(\tau) \\ &= -\frac{1}{12}\omega^2\Delta^3(0) = -\frac{1}{24\omega} + \frac{1}{4}\omega^2\Delta^3(0)\end{aligned} \tag{38}$$

derived partially from Eq. (33).

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On the other hand, the integral (36) can be transformed by the partial integration into another form

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|} = \int d\tau\, \epsilon(\tau)\delta(\tau)\frac{d}{d\tau}\left[-\frac{1}{\lambda}e^{-\lambda|\tau|}\right]$$
$$= \frac{1}{\lambda}\left[2\int d\tau\, \delta^2(\tau)e^{-\lambda|\tau|} + \int d\tau\, \epsilon(\tau)\dot{\delta}(\tau)e^{-\lambda|\tau|}\right], \qquad (39)$$

where the role of the parameter λ becomes crucial. Integrating by parts now the left-hand side of Eq. (32) and using Eq. (39), we obtain

$$\int d\tau\left[\ddot{\Delta}^2(\tau) + 2\omega^2\dot{\Delta}^2(\tau) + \omega^4\Delta^2(\tau)\right] =$$
$$= \int d\tau\, \delta^2(\tau) - \frac{1}{2\omega}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{2\omega}\,. \qquad (40)$$

Contrary to the result (37), it follows from Eqs. (32) and (40) that

$$\int d\tau\, \epsilon^2(\tau)\delta(\tau)e^{-\lambda|\tau|} = 1. \qquad (41)$$

Therefore, in integrals involving singular products of distributions, partial integration is bound to fail since it leaves no room to specify the integral over $\epsilon^2\delta$ consistently. Abandoning the partial integration, we shall define the following rules for an integral over products of two δ-functions in Eqs. (32) and (35)

$$\int d\tau\, f(\tau)\delta^2(\tau) = f(0)\delta(0), \qquad (42)$$

and for the integral over a product of two ϵ-functions with one δ-function in Eq. (33):

$$\int d\tau\, f(\tau)\epsilon^2(\tau)\delta(\tau) = \frac{1}{4}f(0), \qquad (43)$$

for any smooth test function $f(\tau)$.

The obtained relations have reduced all integrals over singular products of correlation functions to regular integrals plus integrals containing $\delta^2(\tau)$ and $\epsilon^2(\tau)\delta(\tau)$. For the last, we have the integration rules (42) and (43) which determine completely the right-hand sides of relations (32), (33) and (35). We are now going to show that these rules hold the required coordinate independence of the path integral up to a three-loop approximation.

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5 Reparametrization Invariance

To first order in g, the sum of Feynman diagrams (17) vanishes:

 $+\,\omega^2$  $-\,\delta(0)$  $=0.$ (44)

The analytic form of this relation is

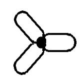
$$\left[-\ddot{\Delta}(0)+\omega^2\Delta(0)-\delta(0)\right]\Delta(0)=0, \tag{45}$$

and the vanishing is a direct consequence of the field equation (27) for the correlation function at the origin.

At order g^2, the same equation reduces the sum of all local diagrams in (18) to a finite result plus a term proportional to $\delta(0)$:

$$\left[-3\left(\frac{1}{2}+a\right)\ddot{\Delta}(0)+15\left(\frac{1}{18}+\frac{a}{5}\right)\omega^2\Delta(0)\right.$$
$$\left.-3\left(a-\frac{1}{2}\right)\delta(0)\right]\Delta^2(0)=\left[3\delta(0)-\frac{2}{3}\,\omega^2\Delta(0)\right]\Delta^2(0). \tag{46}$$

Representing the right-hand side diagrammatically, we obtain the identity

$\Sigma\,(18)=3\delta(0)$  $-\,\frac{2}{3}\,\omega^2$ , (47)

where $\Sigma\,(18)$ denotes the sum of all diagrams in Eq. (18). Using the identity (24) together with the field equation (27), we reduce the sum (19) of all one- and two-loop bubbles diagrams to terms involving $\delta(0)$ and $\delta^2(0)$:

$$-\frac{1}{2!}\left\{2\delta^2(0)\int d\tau\,\Delta^2(\tau)\right.$$
$$\left.-4\delta(0)\int d\tau\,\left[\Delta(0)\dot{\Delta}^2(\tau)-\ddot{\Delta}(0)\Delta^2(\tau)+2\omega^2\Delta(0)\Delta^2(\tau)\right]\right\}$$
$$=2\Delta^2(0)\,\delta(0)+\delta^2(0)\int d\tau\Delta^2(\tau). \tag{48}$$

Hence we find the diagrammatic identity

$-\frac{1}{2!}\,\Sigma\,(19)=2\delta(0)$  $+\;\delta^2(0)$  . (49)

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Now, the terms accompanying $\delta^2(0)$ turn out to be cancelled by similar terms coming from the sum of all three-loop bubbles diagrams in (20). In fact, the identities (24) and (32) lead to

$$\begin{aligned}
-\frac{1}{2!}\int d\tau\, \Big[& -4\Delta(0)\ddot{\Delta}(0)\dot{\Delta}^2(\tau) + 2\Delta^2(0)\ddot{\Delta}^2(\tau) \\
& + 2\ddot{\Delta}^2(0)\Delta^2(\tau) + 8\omega^2\Delta^2(0)\dot{\Delta}^2(\tau) \\
& - 8\omega^2\Delta(0)\ddot{\Delta}(0)\Delta^2(\tau) + 8\omega^4\Delta^2(0)\Delta^2(\tau)\Big] \\
= \; - & \left[\int d\tau\, \delta^2(\tau) + 2\delta(0)\right] \Delta^2(0) - \delta^2(0) \int d\tau\, \Delta^2(\tau). \qquad (50)
\end{aligned}$$

Thus, using the rule (42), we find the diagrammatical sum for all bubbles diagrams:

$$-\frac{1}{2!}\,\Sigma\,(19) - \frac{1}{2!}\,\Sigma(20) = -\delta(0)$$  $\quad . \qquad (51)$

Finally, the relations (25), (26), (33), (34), and (35) reduce the sum (21) of all watermelon-like diagrams to a finite contribution plus integrals involving $\delta^2(\tau)$ and $\epsilon^2(\tau)\delta(\tau)$:

$$\begin{aligned}
-\frac{4}{2!}\int d\tau\, \Big[& \Delta^2(\tau)\ddot{\Delta}^2(\tau) + 4\Delta(\tau)\dot{\Delta}^2(\tau)\ddot{\Delta}(\tau) \\
& + \dot{\Delta}^4(\tau) + 4\omega^2\Delta^2(\tau)\dot{\Delta}^2(\tau) + \frac{2}{3}\omega^4\Delta^4(\tau)\Big] \\
= \; - \; & 2\int d\tau\, \Delta^2(\tau)\delta^2(\tau) + \frac{1}{\omega}\int d\tau\, \epsilon^2(\tau)\delta(\tau) - \frac{4}{3}\,\omega^2\Delta^3(0)\,. \qquad (52)
\end{aligned}$$

Combining these with all local diagrams (47), we easily verify that all finite contributions cancel each other, thus leading, with the help of Eq. (43), to the diagrammatic identity

$$\Sigma\,(18) - \frac{4}{2!}\Sigma(21) = \delta(0)$$  $\quad . \qquad (53)$

The remaining singular terms in Eqs. (51) and (53) are summing up to zero, as required by the coordinate invariance of perturbatively defined path integrals.

The procedure can easily be continued to higher-loop diagrams to obtain integrals over any desired products of singular correlation functions, and over products of δ-functions. At no place we have to specify the value of $\delta(0)$ and the regularization scheme.

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6 Summary

In this note, using the simple rules for relating singular to regular Feynman integrals found in Ref. [3], we discuss the reparametrization invariance of the perturbatively defined path integral for a particle in one-dimensional target space by performing an explicit three-loop calculation of the ground-state energy. The unique rules supply both kinds of cancellation: the loop divergences as well as the finite contributions, thus providing the UV-finiteness simultaneously with the coordinate independence of the quantum mechanical path integrals. This approach has an important advantage allowing us to avoid the explicit calculation of dimensionally regularized integrals over products of distributions. The procedure is *independent* of regularization prescriptions, using only the fact that regularized integrals can be integrated by parts. The results are, of course, perfectly compatible with those derived before in Refs. [1,2] by dimensional regularization.

Considerations of this paper are more general than those of Ref. [5] where two model Hamiltonians related to some transformation were found. It was shown that they lead to the same ground-state energy.

To proof the coordinate independence of perturbatively defined path integrals in one-dimensional target space there is no need for extra compensating noncovariant terms found necessary in the treatment of phase-space path integrals in Refs. [6–8]. A perturbative definition of the phase-space path integral was recently developed in Ref. [9].

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